Math 341 Final Exam Preparation Sheet
Supplement

1. Recall and apply properties of convergence for power series. For a power series $g(x) = \sum_{n=0}^{\infty} a_n x^n$, we know that

(a) if it converges at x_0, then it converges absolutely on $|x| < |x_0|$,
(b) if it converges absolutely at x_0, then it converges uniformly on $[-|x_0|, |x_0|]$,
(c) If it converges at $x = R > 0$, then it converges uniformly on $[0, R]$ (similar result if $x = R < 0$), and
(d) if it converges on a set A, then it converges uniformly on any compact subset K of A.

Exercises. Decide which of the following statements are true or false.

(a) The power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ converges absolutely on $[-2, 2]$.

(b) The power series $\displaystyle\sum_{n=0}^{\infty} \frac{nx^n}{n^2+1}$ converges uniformly on $[-1, 1]$.

(c) The power series $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^3+1}$ converges on every compact subset of $[-1, 1]$.

2. Find the interval where a power series converges. For a power series $g(x) = \sum_{n=0}^{\infty} a_n x^n$ we compute the radius of convergence

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

If $R = 0$ the power series converges only at $x = 0$. If $R = \infty$, then the power series converges at every $x \in \mathbb{R}$. If $0 < R < \infty$, then the interval of convergence for the power series is one of four intervals $(-R, R)$, $[-R, R)$, $(-R, R]$, or $[-R, R]$. The absolute convergence test and the alternating series test are typically used to determine if the power series converges at the endpoints.

Exercises. Decide which of the following statements are true or false.

(a) The interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 x^n}{n!}$ is $(-\infty, \infty)$.

(b) The interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{3nx^n}{n^2+n}$ is $[-1, 1]$

3. Compute a Taylor series for a given infinitely differentiable function. There are two ways of doing this. First is by recognizing the infinitely differentiable function $f(x)$ as

the derivative or integral of $1/(1-x)$ where x can be substituted with another function of x. Second, is by Taylor's Formula

$$a_n = \frac{f^{(n)}(0)}{n!},$$

and then using Lagrange's Remainder Theorem to show that the Taylor series converges to the original function. Sometimes one can do it both ways, giving one as a means to verify the other.

Exercises. Decide which of the following statements are true or false.

(a) The Taylor series of $f(x) = \ln(1+x^2)$ is (i) $x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots$, and (ii) this Taylor series converges to $f(x)$ on $(-1,1)$

(b) The Taylor series of $g(x) = \exp(x^2)$ is (i) $1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \cdots$, and (ii) this Taylor series converges to $g(x)$ on $(-\infty, \infty)$.

4. <u>Construct and refine partitions, and compute upper and lower sums.</u> The choice of partition can give a good estimate of the value of the integral of an integrable function, or show that the upper sums do not get close to the lower sums.

Exercises. Decide which of the following statements are true or false.

(a) We have $\int_0^2 f = 4$ for

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < 1, \\ 2 & \text{if } x = 1, \\ 3 & \text{if } 1 < x \le 2. \end{cases}$$

(b) There exists a function $g : [0,1] \to \mathbb{R}$ such that $U(g) = 1/2$ and $L(g) = 0$.

(c) For $n \in \mathbb{N}$ define $h(x) = (-1)^n$ for $x \in (1/(n+1), 1/n)$ and $h(x) = 0$ for $x = 1/n$ or $x = 0$ or $x = 1$. (The function h has value -1 on the interval $(1/2, 1)$, value 1 on the interval $(1/3, 1/2)$, etc.) Then (i) h is integrable and (ii) $\int_0^1 h = 0$.

5. <u>Recall and apply the properties of the integral.</u> All of the familiar and less familiar rules for integration are too many to list here. Please see the text for all of them.

Exercises. Decide which of the following statements are true or false.

(a) Let $f : [a,b] \to \mathbb{R}$ is bounded: there exists $M > 0$ such that $|f(x)| \le M$ for all $x \in [a,b]$. If f is integrable, then the absolute value of the average value of f is no bigger than M.

(b) The function $g(x) = \sin(1/x)$ if $x \in (0,1]$ and $g(0) = 0$ is integrable on $[0,1]$.