## Math 341 Lecture #2§1.2: Some Preliminaries

We review some preliminary material such as notation, basic operations and concepts, to be used throughout the semester.

Sets. Intuitively speaking, a *set* if a collection of objects, and the objects are referred to as the *elements* of the set.

When x is an element of a set A we write  $x \in A$ , and when x is not an element of A we write  $x \notin A$ .

The union of two sets A and B is

$$A \cup B = \{ x : x \in A \text{ or } x \in B \}.$$

The intersection of two sets A and B is

$$A \cap B = \{ x : x \in A \text{ and } x \in B \}.$$

A set A is a subset of a set B if every element of A is an element of B; we write  $A \subseteq B$ . We say two sets A and B are equal when  $A \subseteq B$  and  $B \subseteq A$ ; we write A = B. Example. For each positive integer n let

$$A_n = \{n, n+1, n+2, \dots\}.$$

These *indexed* sets satisfy the nesting

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq \cdots$$

What then is

$$\bigcup_{n=1}^{\infty} A_n?$$

It is  $A_1$  because of the nesting.

What is

$$\bigcap_{n=1}^{\infty} A_n?$$

It is the empty set  $\emptyset$  because if there were a positive integer m in the intersection, then m would be in each  $A_n$ , but his fails when n > m.

For a subset A of a universal set  $\mathcal{U}$ , the *complement* of A is  $\mathcal{U} - A$ .

Notationally you saw this in Math 290 as  $\overline{A}$ , but here in Math 341 we will use the notation  $A^c$  instead (as the notation  $\overline{A}$  will be used for the "closure" of a set, a notion in topology).

You might remember De Morgan's Laws for the interaction of unions and intersections with complements:

$$(A \cup B)^c = A^c \cap B^c, \ (A \cap B)^c = A^c \cup B^c.$$

These properties extend to indexed sets  $A_i, i \in I$ :

$$\left(\bigcup A_i\right)^c = \bigcap A_i^c, \quad \left(\bigcap A_i\right)^c = \bigcup A_i^c.$$

Functions. For two nonempty sets A and B, a function from A to B is a relation  $f \subseteq A \times B$  that satisfies the "vertical line test" of for each  $a \in A$  there is exactly one ordered pair of the form (a, b) in f.

Notationally we write  $f : A \to B$ , and we write f(a) = b when  $(a, b) \in f$ .

The set A is called the *domain* of f and the set B is called the *codomain* of f.

The range of f is the subset  $f(A) = \{b \in B : b = f(a) \text{ for some } a \in A\}$  of the codomain B.

A function  $f: A \to B$  is *injective* or *one-to-one* if when f(x) = f(y) then x = y.

A function  $f : A \to B$  is surjective or onto if for every  $b \in B$  there is  $a \in A$  such that f(a) = b.

Not all functions come with "nice" algebraic formulas as the unruly function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}, \end{cases}$$

proposed by Dirichlet in 1829 suggests.

The domain of f is  $\mathbb{R}$ , the codomain is  $\mathbb{R}$ , and the range is the finite set  $\{0, 1\}$ .

This function is not injective nor surjective.

What would the graph of this f look like on a computer?

On the other hand, the absolute value function

$$|x| = \begin{cases} x & \text{if } x \ge 0, \\ -x & \text{if } x < 0, \end{cases}$$

is of vast importance in analysis.

Two important properties of |x| are

- (a) |ab| = |a| |b|, and
- (b)  $|a+b| \le |a| + |b|$  (the triangle inequality).

You have it as a homework problem 1.2.6 is provide a proof of the triangle inequality and its companion inequality

$$|a| - |b|| \le |a - b|.$$

Logic and Proofs. To illustrate the importance of the absolute value function, we consider the following which uses the absolute value function to characterize when two real numbers are the same.

Theorem 1.2.6. Two real numbers a and b are equal if and only if for every real number  $\epsilon > 0$  we have that  $|a - b| < \epsilon$ .

Proof. We first show the  $\Rightarrow$  direction: if a = b then for every  $\epsilon > 0$  we have  $|a - b| < \epsilon$ . When a = b we have a - b = 0 so that |a - b| = 0 which is smaller than  $\epsilon$  for all  $\epsilon > 0$ . Now we show the  $\Leftarrow$  direction: if for every real  $\epsilon > 0$  we have  $|a - b| < \epsilon$ , then a = b. Let's prove this by the *contrapositive*, that is, we show that if  $a \neq b$ , then there exists  $\epsilon > 0$  such that  $|a - b| \ge \epsilon$ .

How do we pick the value of  $\epsilon$ ? One choice is  $\epsilon = |a - b|$ , for then  $|a - b| \ge \epsilon$ . Induction. The use of induction arguments will appear with some frequency.

Example. Let  $x_1 = 1$  and for each  $n \in \mathbb{N}$  recursively define a sequence  $\{x_n\}$  by

$$x_{n+1} = (1/2)x_n + 1.$$

Computing the first few values of  $x_n$  we get

$$x_1 = 1, \ x_2 = 3/2, \ x_3 = 7/4.$$

It appears that  $x_1 \leq x_2 \leq x_3$ , and we are tempted to claim that  $x_n \leq x_{n+1}$ .

To prove this claim we will induction.

We have already established the first or base step:  $x_1 \leq x_2$ .

We now assume for the second step that  $x_n \leq x_{n+1}$  for some *n* (the induction hypothesis), and use this to prove that  $x_{n+1} \leq x_{n+2}$ .

Multiplying our assumption  $x_n \leq x_{n+1}$  on both sides by 1/2 and adding 1 to both sides gives

$$(1/2)x_n + 1 \le (1/2)x_{n+1} + 1$$

By our definition of  $x_{n+1} = (1/2)x_n + 1$  we obtain

$$x_{n+1} \le x_{n+2}.$$

By induction the claim is established for all  $n \ge 1$ .

If the sequence  $\{x_n\}$  converged, what would be its limit? If

$$\lim_{n \to \infty} x_n = L,$$

then

$$\lim_{n \to \infty} x_{n+1} = L$$

as well, so that  $x_{n+1} = (1/2)x_n + 1$  would give L = (1/2)L + 1, which solves to give (1/2)L = 1, or L = 2.

How do we establish that  $\{x_n\}$  converges? The Axiom of Completeness for  $\mathbb{R}$ .