## Math 341 Lecture \#3 <br> §1.3: The Axiom of Completeness

We have seen that $\sqrt{2}$ is a "gap" in $\mathbb{Q}$ (Theorem 1.1.1).
We think of $\mathbb{Q}$ as a subset of $\mathbb{R}$ and that $\mathbb{R}$ has no "gaps."
This accepted assumption about $\mathbb{R}$ is known as the Axiom of Completeness: Every nonempty set of real numbers that is bounded above has a least upper bound.
When one properly "constructs" the real numbers from the rational numbers, one can prove that the Axiom of Completeness as a theorem.
This proof is rather lengthy; an outline of it given at the end of the text.
Almost everything in Calculus is a consequence of the Axiom of Completeness, as we shall see.
We begin by defining the notions of bounded above and least upper bound used in the Axiom of Completeness.
Definition 1.3.1. A set $A \subseteq \mathbb{R}$ is bounded above if there exists a $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$.
The number $b$ is called an upper bound for $A$.
Similarly, the set $A$ is bounded below if there exists an $l \in \mathbb{R}$ such that $l \leq a$ for all $a \in A$, and $l$ is called a lower bound for $A$.
Definition 1.3.2. An $s \in \mathbb{R}$ is a least upper bound for a set $A \subseteq \mathbb{R}$ if
(i) $s$ is an upper bound for $A$, and
(ii) if $b$ is an upper bound for $A$, then $s \leq b$.

Another name for a least upper bound is supremum; this is written

$$
s=\sup A \text {. }
$$

A greatest lower bound or infimum is similarly defined and is denoted by $\inf A$.
Uniqueness of Supremum. A set $A$ can have many upper bounds, but only one least upper bound.
The argument for uniqueness is to assume that there are two least upper bounds $s_{1}$ and $s_{2}$ of a set $A$.
Then by the second property of a least upper bound, we have $s_{1} \leq s_{2}$ and we have $s_{2} \leq s_{1}$.
This implies that $s_{1}=s_{2}$.
Example 1.3.3. What is the supremum and infimum of

$$
A=\left\{\frac{1}{n}: n \in \mathbb{N}\right\}=\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\} ?
$$

Numbers such as 2 and $3 / 2$ are upper bounds for $A$, but the supremum should be 1 .

The number 1 is an upper bound for $A$ because $1 / n \leq 1$ for all $n \in \mathbb{N}$.
If $b$ is another upper bound for $A$, then $1 \leq b$, but this is property (ii) of a supremum, and so $\sup A=1$.
It appears that $\inf A=0$, but showing this requires tools we do not have yet.
Does $A$ have a maximum value? a minimum value?
Definition 1.3.4. An $a_{0} \in \mathbb{R}$ is a maximum of a set $A$ if $a_{0} \in A$ and $a \leq a_{0}$ for all $a \in A$.
Similary, an $a_{1} \in \mathbb{R}$ is a minimum value of $A$ if $a_{1} \in A$ and $a_{1} \leq a$ for all $a \in A$.
Examples. (a) The set $A$ in Example 1.3.3 has a maximum value of 1 but does not have a minimum value.
(b) The interval $[0,1$ ) has a minimum value of 0 but does not have a maximum value, although its supremum is 1 .
Example 1.3.6. Consider the set

$$
S=\left\{r \in \mathbb{Q}: r^{2}<2\right\}
$$

and pretend for a moment that our world consists only of rational numbers.
The set $S$ is bounded above: numbers like 3, 2, and 1.5 are upper bounds,
But what happens when we look for the least upper bound of $S$ ?
The decimal expansion for $\sqrt{2}$ is $1.4142 \ldots$, and so other upper bounds for $S$ are 142/100, 1415/100, etc.
Among the rational numbers there is no least upper bound: $\sqrt{2} \notin \mathbb{Q}$ by Theorem 1.1.1!
Every time we think we have found the supremum we can find another upper bound that is rational and smaller.
Among the reals, we would find that $\sup S=\sqrt{2}$.
The Axiom of Completeness asserts that such as number as $\sqrt{2}$ exists!
Example 1.3.7. Here is a property of the supremum.
Let $A \subseteq \mathbb{R}$ be a nonempty set which is bounded above.
For $c \in \mathbb{R}$ define

$$
c+A=\{c+a: a \in A\} .
$$

Then

$$
\sup (c+A)=c+\sup A
$$

To prove this we have to verify the two properties of a supremum for the set $c+A$.
Set $s=\sup A$.
Then $a \leq s$ for all $a \in A$.
Hence $a+c \leq a+s$ for all $a \in A$.
This means that $a+s$ is an upper bound for $c+A$.

Now let $b$ be an arbitrary upper bound for $c+A$, so that $c+a \leq b$ for all $a \in A$.
This is equivalent to $a \leq b-c$ for all $a \in A$.
Thus $b-c$ is an upper bound for $A$.
Because $s$ is the least upper bound for $A$ we have that $s \leq b-c$.
Hence $s+c \leq b$, making $s+c$ the least upper bound for $c+A$.
Another useful tool in working with a supremum is the following result.
Lemma 1.3.8. Assume that $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then $s=\sup A$ if and only if for every real $\epsilon>0$ there exists $a \in A$ such that $s-\epsilon<a$.

Proof. First we show the direction $\Rightarrow$.
Suppose that $s=\sup A$.
For an arbitrary $\epsilon>0$ consider $s-\epsilon$.
Because $s$ is the least upper bound, then $s-\epsilon$ is not an upper bound.
Then there must be an $a \in A$ such that $s-\epsilon<a$, for otherwise, $s-\epsilon$ would be an upper bound.

Next we show the direction $\Leftarrow$.
Suppose for the upper bound $s$ we have that for every $\epsilon>0$ the existence of $a \in A$ such that $s-\epsilon<a$.

To show that $s$ is the least upper bound, we need to show for any upper bound $b$ we have $s \leq b$.
Suppose to the contrary, that an upper bound $b$ satisfies $s>b$.
Then for $\epsilon=s-b>0$ there is $a \in A$ such that $s-\epsilon<a$.
Since $\epsilon=s-b$ the inequality $s-\epsilon<a$ becomes $s-(s-b)<a$ or $b<a$, that is there is $a \in A$ such that $b<a$, meaning $b$ is not an upper bound.

This contradiction implies that $s \leq b$, and hence that $s$ is the least upper bound.

