Math 341 Lecture #4
§1.4: Consequences of Completeness

We are going to see how the Axiom of Completeness shows there are no "gaps" in $\mathbb{R}$.

Theorem 1.4.1 (Nested Interval Property). For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] \subset \mathbb{R}$. Assume also that $I_n \supseteq I_{n+1}$. Then the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots,$$

has a nonempty intersection:

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

Proof. Consider the set of left endpoints of the intervals I_n:

$$A = \{a_n : n \in \mathbb{N}\}.$$

Because of the nesting of the intervals, every right endpoint b_n is an upper bound of A.

By the Axiom of Completeness, we can set

$$x = \sup A.$$

Because x is an upper bound for A, then $a_n \leq x$ for all n.

Also, because each b_n is an upper bound for A, we have $x \leq b_n$.

These imply that $a_n \leq x \leq b_n$, i.e., $x \in I_n$, for all $n \in \mathbb{N}$.

Thus $x \in \cap_{n=1}^{\infty} I_n$ and the intersection is not empty. $\square$

How does $\mathbb{N}$ sit inside $\mathbb{R}$?

Theorem 1.4.2 (Archimedean Property). (i) Given any $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$. (ii) Given any positive $y \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $1/n < y$.

Proof. (i) Assume, for a contradiction, that $\mathbb{N}$ is bounded above.

Then by the Axiom of Completeness, the number $\alpha = \sup \mathbb{N}$ exists.

The number $\alpha - 1$ is not an upper bound (by Lemma 1.3.8 with $\epsilon = 1$), and so there is an $n \in \mathbb{N}$ such that $\alpha - 1 < n$, or

$$\alpha < n + 1.$$

But $n + 1 \in \mathbb{N}$, implying that α is not an upper bound for $\mathbb{N}$.

Part (ii) follows by letting $x = 1/y$. $\square$

How does $\mathbb{Q}$ sit inside $\mathbb{R}$?

Theorem 1.4.3 (Density of $\mathbb{Q}$ in $\mathbb{R}$). For every two real numbers a and b with $a < b$, there exists an $r \in \mathbb{Q}$ such that $a < r < b$.

Proof. To keep things simple, assume that $0 \le a < b$. [The case where $a < 0$ is handed similarly.]

We want to find $m, n \in \mathbb{N}$ with $n \neq 0$ such that $a < m/n < b$.

By Theorem 1.4.2 (Archimedean Property), we may pick $n \in \mathbb{N}$ large enough (so $n \neq 0$) such that
$$\frac{1}{n} < b - a.$$
This can be rewritten as
$$a < b - \frac{1}{n}.$$
Having chosen n we now choose $m \in \mathbb{N}$ so that
$$m - 1 \le na < m.$$
The second inequality yields
$$a \le \frac{m}{n},$$
while the first inequality yields
$$m \le na + 1.$$
Our choice for n this implies that
$$m \le n\left(b - \frac{1}{n}\right) + 1 = nb - 1 + 1 = nb.$$
This implies that $m/n < b$. $\square$

We will let $\mathbb{I} = \mathbb{R} - \mathbb{Q}$ denote the set of irrational numbers.

Corollary 1.4.4 (Density of $\mathbb{I}$ in $\mathbb{R}$). Given $a, b \in \mathbb{R}$ with $a < b$, there exists an irrational number t satisfying $a < t < b$.

The proof of this a homework problem 1.4.5.

We showed in Theorem 1.1.1 that there is no rational number whose square is 2, or equivalently, that $\sqrt{2}$ is not a rational number.

We will prove there is a real number whose square is 2.

Theorem 1.4.5. There exists a real number $\alpha \in \mathbb{R}$ satisfying $\alpha^2 = 2$.

Proof. We consider the set
$$T = \{t \in \mathbb{R} : t^2 < 2\}.$$
This is similar to what we saw in Example 1.3.6 (wherein we only considered rational numbers satisfying $t^2 < 2$ and showed that the set of such t does not have a least upper bound as a rational number).

Let $\alpha = \sup T$.

We will show that $\alpha^2 = 2$ by eliminating the possibilities $\alpha^2 > 2$ and $\alpha^2 < 2$ by way of contradiction.

We will see that we will violate one of the two properties of a supremum in each case.

Suppose $\alpha^2 < 2$.

For an integer $n \geq 2$ we have

$$\begin{aligned}\left(\alpha + \frac{1}{n}\right)^2 &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \\ &< \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} \\ &= \alpha^2 + \frac{2\alpha + 1}{n}.\end{aligned}$$

Because $\alpha^2 < 2$ we can make $\alpha^2 + (2\alpha + 1)/n < 2$ by choosing n large enough.

Specifically we choose $n_0 \geq 2$ large enough so that

$$\frac{1}{n_0} < \frac{2 - \alpha^2}{2\alpha + 1}.$$

(We got this choice of n_0 by solving $\alpha^2 + (2\alpha + 1)/n < 2$ for $1/n$.)

This implies that

$$\frac{2\alpha + 1}{n_0} < 2 - \alpha^2,$$

so that

$$\left(\alpha + \frac{1}{n_0}\right)^2 < \alpha^2 + (2 - \alpha^2) = 2.$$

Thus the real number $\alpha + 1/n_0$ belongs to T but is bigger than its supremum α, a contradiction to α being an upper bound of T.

Thus $\alpha^2 < 2$ is impossible.

Now suppose $\alpha^2 > 2$, and this time for an integer $n \geq 1$ we have

$$\begin{aligned}\left(\alpha - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n}.\end{aligned}$$

The remainder of the proof is a homework problem 1.4.7.

(Reach a statement that contradicts that α is the least of the upper bounds of T.) $\square$