Math 341 Lecture #5
§1.5: Cardinality

The term "cardinality" is one way to assess the size of a set.

A set A has *finite cardinality* if $A = \emptyset$ or $A = \{a_1, a_2, \ldots, a_n\}$ for some $n \in \mathbb{N}$.

A set is called *infinite* if it is not finite.

Recall from Math 290 that we use functions to compare cardinalities of sets.

Definition 1.5.1. A function $f : A \to B$ is *injective* or *one-to-one* (1-1) if $a_1 \neq a_2$ in A implies $f(a_1) \neq f(a_2)$) in B, i.e., if $f(a_1) = f(a_2)$ implies $a_1 = a_2$.

A function f is *surjective* or *onto* if for any given $b \in B$ there is an $a \in A$ such that $f(a) = b$.

A function $f : A \to B$ is a *bijection* (or a one-to-one correspondence) if f is one-to-one and onto.

Definition 1.5.2. Two sets A and B have the *same cardinality* if there exists a bijection $f : A \to B$, and we write $A \sim B$.

Example 1.5.3. (i) Does the set of positive odd integers have the same cardinality as $\mathbb{N}$?

The set of positive odd integers is

$$O = \{1, 3, 5, 7, \ldots\}.$$

The function $f : \mathbb{N} \to O$ defined by

$$f(n) = 2n - 1$$

is injective because

$$f(n) = f(m) \ \Rightarrow\ 2n - 1 = 2m - 1 \ \Rightarrow\ 2n = 2m \ \Rightarrow\ n = m,$$

and this function is surjective because each positive odd integer is of the form $2n - 1$ for some $n \in \mathbb{N}$.

Thus $O \sim \mathbb{N}$ (although O is only half of the set $\mathbb{N}$, but this is what can happen with infinite sets).

(ii) It is true that $\mathbb{N} \sim \mathbb{Z}$?

To show this we have to construct a bijection $f : \mathbb{N} \to \mathbb{Z}$.

For this try the function

$$f(n) = \begin{cases} (n-1)/2 & \text{if } n \text{ is odd,} \\ -n/2 & \text{if } n \text{ is even.} \end{cases}$$

A careful investigation shows that $f(1) = 0$, $f(2) = -1$, $f(3) = 1$, $f(4) = -2$, $f(5) = 2$, etc., so that f is a bijection.

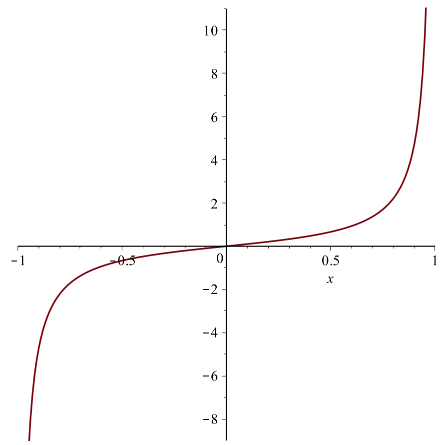
Example 1.5.4. Is $(-1,1) \sim \mathbb{R}$?

To show that is true we need to find a bijection $f : (-1,1) \to \mathbb{R}$.

Here is one such bijection:

$$f(x) = \frac{x}{1-x^2}.$$

Here is its graph.



By Calculus we know this function is injective ($f'(x) > 0$ for all $x \in (-1,1)$, and is surjective (vertical asymptotes at $x = -1$ and $x = 1$).

Definition 1.5.5. A set A is *countable* (or countable infinite) if $A \sim \mathbb{N}$.

An infinite set that is not countable is called an *uncountable* set.

Example 1.5.3 (ii) shows that $\mathbb{Z}$ is a countable set.

Theorem 1.5.6. (i) $\mathbb{Q}$ is countable. (ii) $\mathbb{R}$ is uncountable.

Proof. (i) Set $A_1 = \{0\}$, and for each integer $n \geq 2$ define

$$A_n = \left\{ \pm\frac{p}{q} : \text{ where } p, q \in \mathbb{N} \text{ are in lowest terms with } p + q = n \right\}.$$

Then

$$\begin{aligned}
A_2 &= \left\{ \pm\frac{1}{1} \right\}, \\
A_3 &= \left\{ \pm\frac{1}{2}, \pm\frac{2}{1} \right\}, \\
A_4 &= \left\{ \pm\frac{1}{3}, \pm\frac{3}{1} \right\}, \\
A_5 &= \left\{ \pm\frac{1}{4}, \pm\frac{2}{3}, \pm\frac{3}{2}, \pm\frac{4}{1} \right\}, \\
&\text{etc.}
\end{aligned}$$

The crucial observations are that each A_n is a finite set and each rational number appears in exactly one of these sets.

We construct a bijection $f : \mathbb{N} \to \mathbb{Q}$ by associating the integer 1 with the single element of A_1, the integers 2 and 3 with the two elements of A_2, the integers 4, 5, 6, and 7 with the four elements of A_3, etc.

(ii) To show that $\mathbb{R}$ is uncountable, we assume to the contrary that there is a bijection $f : \mathbb{N} \to \mathbb{R}$, and reach a contradiction.

(We will use the Nested Interval Property to reach a contradiction, and not Cantor's diagonalization argument which we will review next time.)

Then we can enumerate the real numbers; $\mathbb{R} = \{x_1, x_2, x_3, \ldots\}$.

Let I_1 be a closed interval that does not contain x_1, and let I_2 be a closed subinterval of I_1 that does not contain x_2.

The existence of I_2 follows because I_1 certainly contains two disjoint closed subintervals and x_2 can only be in one of them.

Continuing choosing closed intervals I_n such that

(i) $I_{n+1} \subseteq I_n$, and
(ii) $x_{n+1} \notin I_{n+1}$.

For a real number x_{n_0} in our enumerated list of $\mathbb{R}$, we have that $x_{n_0} \notin I_{n_0}$, and so

$$x_{n_0} \notin \bigcap_{n=1}^{\infty} I_n.$$

Since we have enumerated all the real numbers, then

$$\bigcap_{n=1}^{\infty} I_n = \emptyset.$$

However, the Nested Interval Property (Theorem 1.4.1) guarantees that

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

This contradiction implies that there is no bijection $f : \mathbb{N} \to \mathbb{R}$, so $\mathbb{R}$ is uncountable. □

Theorem 1.5.7. If $A \subseteq B$ and B is countable, then A is either countable or finite.

The proof of this is a homework problem 1.5.1.

Theorem 1.5.8. (i) If $A_1, A_2, \ldots, A_m$ are each countable sets, then $A_1 \cup A_2 \cup \cdots \cup A_m$ is countable. (ii) If A_n is a countable set for each $n \in \mathbb{N}$, then $\cup_{n=1}^{\infty} A_n$ is countable.

The proofs of these are a homework problem 1.5.3.