Math 341 Lecture #7
§2.1, 2.2: Series, Sequences

For finite sums we have the commutative and associative properties holding, but what about infinite sums?

Example. For positive integers i and j, consider the numbers

$$a_{ij} = \begin{cases} 2^{i-j} & \text{if } j > i, \\ -1 & \text{if } i = j, \\ 0 & \text{if } j < i. \end{cases}$$

We can visualize these numbers by a grid:

$$\begin{bmatrix} -1 & 1/2 & 1/4 & 1/8 & 1/16 & \dots \\ 0 & -1 & 1/2 & 1/4 & 1/8 & \dots \\ 0 & 0 & -1 & 1/2 & 1/4 & \dots \\ 0 & 0 & 0 & -1 & 1/2 & \dots \\ 0 & 0 & 0 & 0 & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

What happens when we add the rows first? We get

$$\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{ij} \right) = \sum_{i=1}^{\infty} 0 = 0$$

because $1/2+1/4+1/8+\cdots = 1$ (geometric series for $r = 1/2$ with first term 1 missing).

What happens when we add the columns first? We get

$$\sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} a_{ij} \right) = -1 - 1/2 - 1/4 - 1/8 - \cdots = -2$$

because $1 + 1/2 + 1/4 + 1/8 + \cdots = 2$ (geometric series for $r = 1/2$).

This shows that commutativity of addition can fail for infinite sums.

Example. Consider the series

$$\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots.$$

Associating the sum in one way gives

$$(-1 + 1) + (-1 + 1) + (-1 + 1) + \cdots = 0 + 0 + 0 + \cdots = 0,$$

while associating the sum in another way gives

$$-1 + (1 - 1) + (1 - 1) + (1 - 1) + \cdots = -1 + 0 + 0 + \cdots = -1.$$

This shows that associativity of addition can fail for infinite sums.

An understanding of series depends heavily upon an understanding of sequences and their convergence or divergence.

Definition 2.2.1. A *sequence* is a function whose domain is $\mathbb{N}$.

We typically write a sequence as $(a_n)_{n=1}^{\infty}$, or simply (a_n) where $n \in \mathbb{N}$ is implicitly understood.

Examples.

(a) $(2^{-n+1})_{n=1}^{\infty} = \left(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots\right)$.

(b) $\left(\frac{1+n}{n}\right) = \left(2, \frac{3}{2}, \frac{4}{3}, \cdots\right)$.

Definition 2.2.3. A sequence (a_n) *converges* to a real number a if, for every positive $\epsilon > 0$, there exists a positive integer N such that whenever $n \geq N$ it follows that $|a_n - a| < \epsilon$.

Some notations for a convergence sequence (a_n) are

$$\lim_{n\to\infty} a_n = a, \ \lim a_n = a, \ (a_n) \to a.$$

The notion of $|a_n - a| < \epsilon$ requires some special attention.

Definition 2.2.4. Given $a \in \mathbb{R}$ and a positive $\epsilon > 0$, the set

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

is called the ϵ-neighbourhood of a.

This set $V_\epsilon(a)$ is an open interval with a at its center and with a "radius" of ϵ.

This notion of a neighbourhood leads to a "topological" version of convergence.

Definition 2.2.3B. A sequence (a_n) converges to a if, given any ϵ-neighbourhood $V_\epsilon(a)$, there exists a point (a.k.a. N) in the sequence after which all of the terms of the sequence are in $V_\epsilon(a)$.

This says that only finitely many terms of the sequence are not in $V_\epsilon(a)$.

The number N is the point where the sequence enters $V_\epsilon(a)$ and never leaves.

You should recognize that the value of N will generally depend on the choice of ϵ: the smaller ϵ, the bigger the value of N for which the sequence enters $V_\epsilon(a)$ never to leave.

Usually the choice of N can be determined by how the terms in the sequence (a_n) are defined by n.

Example 2.2.5. Consider (a_n) for $a_n = 1/\sqrt{n}$.

As n gets bigger (i.e., approaches ∞), the value of a_n approaches 0, and we "conclude" that

$$\lim \frac{1}{\sqrt{n}} = 0.$$

To prove this rigourously we need to understand the relationship between the choice of ϵ and the value of N needed to have $a_n \in V_\epsilon(0)$ for all $n \geq N$.

If we take $\epsilon = 1/10$, then we are seeking for a value of N such that

$$\left| \frac{1}{\sqrt{n}} - 0 \right| < \frac{1}{10}.$$

We recognize that with $n = 100$ we get $1/\sqrt{n} = 1/10$, and so we can pick $N = 101$ or any larger integer.

If we take $\epsilon = 1/50$, then we are seeking for a value of N such that

$$\left| \frac{1}{\sqrt{n}} - 0 \right| < \frac{1}{50}.$$

That is we are solving

$$\frac{1}{\sqrt{n}} < \frac{1}{50}$$

for n which gives

$$n > 50^2 = 2500.$$

We can pick $N = 2501$ or any larger integer.

The whole point of this is that no matter small we choose ϵ to be, we can find a value of N for which

$$\left| \frac{1}{\sqrt{n}} - 0 \right| < \epsilon$$

for all $n \geq N$:

$$\frac{1}{\sqrt{n}} < \epsilon \Rightarrow n > \frac{1}{\epsilon^2}.$$

We then have a proof that the sequence converges to 0: for every $\epsilon > 0$ choose $N \in \mathbb{N}$ by

$$N > \frac{1}{\epsilon^2}.$$

Then for all $n \geq N$, we have

$$\left| \frac{1}{\sqrt{n}} - 0 \right| = \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \epsilon.$$

The first inequality follows because $n \geq N$, and the second inequality follows because $N > 1/\epsilon^2$.

Not all sequences converge, like $(a_n) = (-1)^n = (-1, 1, -1, 1, -1, \cdots)$. Why does this not converge?

Definition 2.2.9. A sequence that does not converge is said to *diverge*.