Math 341 Lecture #8
§2.3: The Algebraic and Order Limit Theorems

The point of having the logically tight definition of convergence of a sequence is so that we can *prove* theorems about convergent sequences, not just rely on good guesses.

In a homework problem (Exercise 2.2.6), you will show that the limit of convergent sequence is unique.

Boundedness is another property of convergent sequences.

Definition 2.3.1. A sequence (x_n) is bounded if there is a number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem 2.3.2. Every convergent sequence is bounded.

Proof. Assume that (x_n) is a convergent sequence: there is a unique number l such that

$$\lim_{n\to\infty} x_n = l.$$

Thus for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $|x_n - l| < \epsilon$ for all $n \geq N$.

As this is true for any ϵ, we can take $\epsilon = 1$, and so there is an $N \in \mathbb{N}$ such that $|x_n - l| < 1$ for all $n \geq N$.

This means that $l - 1 < x_n < l + 1$ for all $n \geq N$.

When we think about these inequalities, we can conclude that $|x_n| < |l| + 1$ for all $n \geq N$.

[This is starting to look like (x_n) is bounded.]

But we have not yet accounted for x_n when $1 \leq n \leq N - 1$.

But as there are only finitely many of these, we define

$$M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |l| + 1\}.$$

With this value of M we obtain $|x_n| \leq M$ for all $n \in \mathbb{N}$. □

Convergent sequences behave well (as we would expect) when it comes to scalar multiplication, addition, multiplication, and division.

Theorem 2.3.3 (Algebraic Limit Theorem). If $a = \lim a_n$ and $b = \lim b_n$, then

(i) $\lim(ca_n) = ca$ for all $c \in \mathbb{R}$,

(ii) $\lim(a_n + b_n) = a + b$,

(iii) $\lim(a_n b_n) = ab$,

(iv) $\lim(a_n/b_n) = a/b$ provided $b \neq 0$.

Proof. (i) First consider the case where $c \neq 0$.

We want to show that for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $|ca_n - ca| < \epsilon$ when $n \geq N$.

By a rule of the absolute value we notice that

$$|ca_n - ca| = |c(a_n - a)| = |c|\ |a_n - a|.$$

Here is where we make use of the "for every ϵ" in the definition of convergent sequence.

We take for this "ϵ" the quantity $\epsilon/|c| > 0$, for which there is an $N \in \mathbb{N}$ such that

$$|a_n - a| < \frac{\epsilon}{|c|} \text{ for all } n \geq N.$$

Now we put the pieces together to get

$$|ca_n - ca| = |c|\ |a_n - a| < |c|\ \frac{\epsilon}{|c|} = \epsilon \text{ for all } n \geq N.$$

The case of $c = 0$ is nothing more than showing the constant zero sequence converges to 0.

(ii) We want to show that $|(a_n + b_n) - (a + b)|$ is as small as we want for all large n.

To use the convergence of (a_n) and (b_n) we need to separate a_n and b_n and then use the triangle inequality,

$$\begin{aligned} |(a_n + b_n) - (a + b)| &= |(a_n - a) + (b_n - b)| \\ &\leq |a_n - a| + |b_n - b|. \end{aligned}$$

For $\epsilon > 0$ there exist $N_1, N_2 \in \mathbb{N}$ such that

$$\begin{aligned} |a_n - a| &< \frac{\epsilon}{2} \text{ for all } n \geq N_1, \\ |b_n - b| &< \frac{\epsilon}{2} \text{ for all } n \geq N_2. \end{aligned}$$

Notice that for the "ϵ" we took $\epsilon/2$.

How do we choose the N that goes with ϵ? We take the larger of N_1 and N_2:

$$N = \max\{N_1, N_2\}.$$

With this choose of N we have $n \geq N \geq N_1$ and $n \geq N \geq N_2$, and so for $n \geq N$ we obtain

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

(iii) We want to show that $|a_n b_n - ab|$ is as small as we want for all large n.

This one take some fancier algebra:

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &\leq |a_n b_n - ab_n| + |ab_n - ab| \\ &= |b_n|\ |a_n - a| + |a|\ |b_n - a|. \end{aligned}$$

Let $\epsilon > 0$.

If $a \neq 0$, then since $(b_n) \to b$ there is an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ we have

$$|b_n - b| \leq \frac{1}{|a|}\frac{\epsilon}{2}.$$

The case of $a = 0$ is left to you.

For $|b_n|\ |a_n - a|$, how do we handle the $|b_n|$?

We know that (b_n) converges, and so by Theorem 2.3.2, the sequence (b_n) is bounded: there is a number $M > 0$ such that $|b_n| \leq M$ for all $n \geq 1$.

We can then write $|b_n|\ |a_n - a| \leq M|a_n - a|$ for all $n \in \mathbb{N}$.

Because $(a_n) \to a$, there is an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$ we have

$$|a_n - a| < \frac{1}{M}\frac{\epsilon}{2}.$$

Again, notice the clever choice of "ϵ" here.

We pick $N = \max\{N_1, N_2\}$, so that for all $n \geq N$, we have

$$|a_n b_n - ab| \leq M|a_n - a| + |a|\ |b_n - b| < M\frac{1}{M}\frac{\epsilon}{2} + |a|\frac{1}{|a|}\frac{\epsilon}{2} = \epsilon.$$

The argument for (iv) is left to you to read in the text. It shows that $1/b_n$ converges to $1/b$ and then calls upon (iii). $\square$

Limits of convergent sequences also behave well with respect to the order of $\leq$.

Theorem 2.3.4 (Order Limit Theorem). Suppose that $a = \lim a_n$ and $b = \lim b_n$.

(i) If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $a \geq 0$.

(ii) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.

(iii) If there exists $c \in \mathbb{R}$ for which $c \leq b_n$ for all $n \in \mathbb{N}$, then $c \leq b$. Similiary, if $a_n \leq c$ for all $n \in \mathbb{N}$, then $a \leq c$.

Proof. (i) This will be proved by contradiction: we suppose that $a < 0$ and we will show that $a_n < 0$ for some n, a contradiction.

For $\epsilon = |a|$ there exists $N \in \mathbb{N}$ such that $|a_n - a| < \epsilon = |a|$ when $n \geq N$.

Taking $n = N$, we have $|a_N - a| < |a|$, or unwrapping the absolute value in the middle,

$$-|a| + a < a_N < |a| + a = 0.$$

(ii) The Algebraic Limit Theorem implies that the sequence $(b_n - a_n)$ converges to $b - a$. Because $b_n - a_n \geq 0$ for all $n \in \mathbb{N}$, we apply part (i) to conclude that $b - a \geq 0$.

(iii) Take $a_n = c$ (or $b_n = c$) and apply part (ii). $\square$.