Math 341 Lecture #12
§2.7: Infinite Series

Recall that the convergence of an infinite series $\sum_{k=1}^{\infty} a_k$ is defined in terms of the convergence of the sequence of its partial sums (s_m), where $s_m = \sum_{k=1}^{m} a_k$:

$$\sum_{k=1}^{\infty} a_k = A \text{ means that } \lim_{m\to\infty} s_m = A.$$

Thus we can translate results about convergent sequences to convergent series.

Theorem 2.7.1 (Algebraic Limit Theorem for Series). If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then

(i) $\displaystyle\sum_{k=1}^{\infty} ca_k = cA$ for all $c \in \mathbb{R}$, and

(ii) $\displaystyle\sum_{k=1}^{\infty} (a_k + b_k) = A + B$.

The proof of this is in the Appendix.

Missing from Theorem 2.7.1 is any mention about the product of two series; see Section 2.8 for this.

Theorem 2.7.2 (Cauchy Criterion for Series). The series $\sum_{k=1}^{\infty} a_k$ converges if and only if, for given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n > m \geq N$ it follows that

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon.$$

Proof. Observe for positive integers $n > m$ that

$$|s_n - s_m| = |a_{m+1} + a_{m+2} + \cdots + a_n|.$$

Now apply the Cauchy Criterion for sequences. □

Theorem 2.7.3. If $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$.

The proof of this is in the Appendix.

The converse of Theorem 2.7.3 is false because of the Harmonic series.

The contrapositive of Theorem 2.7.3 – if $(a_k) \not\to 0$, then $\sum_{k=1}^{\infty} a_k$ diverges – gives a divergence test for series.

Theorem 2.7.4 (Comparison Test). Assume (a_k) and (b_k) satisfy $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$.

(i) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

(ii) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

The proof of this is in the Appendix.

The usefulness of the Comparison Test depends on knowing series that converge or diverge.

Recall that we know $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$.

Example 2.7.5 (Geometric Series). When $|r| < 1$, we have that

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$

Theorem 2.7.6 (Absolute Convergence Test). If the series $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

Proof. Suppose $\sum_{k=1}^{\infty} |a_k|$ converges.

By the Cauchy Criterion for Series, for $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$|a_{m+1}| + |a_{m+2}| + \cdots + |a_n| = \Big| \, |a_{m+1}| + |a_{m+2}| + \cdots + |a_n| \, \Big| < \epsilon$$

for all $n > m \geq N$.

By the triangle inequality, we have

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \leq |a_{m+1}| + |a_{m+2}| + \cdots + |a_n|,$$

for all $n > m \geq N$, and so by the Cauchy Criterion for Series, we have that $\sum_{k=1}^{\infty} a_k$ converges. □

The converse of the Absolute Convergence Test is false, as the alternating series

$$\sum_{k=1}^{\infty} (-1)^{k+1}/k$$

demonstrates.

Theorem 2.7.7 (Alternating Series Test). Let (a_n) be a sequence satisfying

(i) $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$, and

(ii) $(a_n) \to 0$.

Then the alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges.

The proof of this is a homework problem 2.7.1.

Definition 2.7.8. We say a series $\sum_{k=1}^{\infty} a_k$ *converges absolutely* if $\sum_{k=1}^{\infty} |a_k|$ converges.

We say a series $\sum_{k=1}^{\infty} a_k$ *converges conditionally* if $\sum_{k=1}^{\infty} a_k$ converges but $\sum_{k=1}^{\infty} |a_k|$ diverges.

We can now address the issue of the order of addition in an infinite series.

Definition 2.7.9. A series $\sum_{k=1}^{\infty} b_k$ is a *rearrangement* of a series $\sum_{k=1}^{\infty} a_k$ if there is a bijection $f : \mathbb{N} \to \mathbb{N}$ such that $b_{f(k)} = a_k$ for all $k \in \mathbb{N}$.

Theorem 2.7.10. If $\sum_{k=1}^{\infty} a_k$ converges absolutely, then any rearrangement of $\sum_{k=1}^{\infty} a_k$ converges to the same limit.

Proof. Assume $\sum_{k=1}^{\infty} a_k$ converges absolutely to A, and let $\sum_{k=1}^{\infty} b_k$ be a rearrangement of $\sum_{k=1}^{\infty} a_k$.

For the partial sums set

$$s_m = a_1 + \cdots + a_m, \;\; t_m = b_1 + \cdots + b_m.$$

We want to show that $(t_m) \to A$.

By the convergence of $\sum_{k=1}^{\infty} a_k$, for $\epsilon > 0$, there exists an $N_1 \in \mathbb{N}$ such that

$$|s_m - A| < \frac{\epsilon}{2} \text{ for all } m \geq N_1.$$

Applying the Cauchy Criterion to the convergent $\sum_{k=1}^{\infty} |a_k|$, there is an $N_2 \in \mathbb{N}$ such that

$$\sum_{k=m+1}^{n} |a_k| < \frac{\epsilon}{2} \;\text{ for all } n > m \geq N_2.$$

Now we take $N = \max\{N_1, N_2\}$.

The finite set of terms $\{a_1, a_2, \ldots, a_N\}$ appears somewhere in the rearranged series $\sum_{k=1}^{\infty} b_k$.

If $f : \mathbb{N} \to \mathbb{N}$ is the bijection satisfying $b_{f(k)} = a_k$, then we can move far enough out in the series $\sum_{k=1}^{\infty} b_k$ to account for $\{a_1, \ldots, a_N\}$ by choosing

$$M = \max\{f(k) : 1 \leq k \leq N\}.$$

This choice of M satisfies $M \geq N$ because $f(k) \geq N$ for some $1 \leq k \leq N$.

For $m \geq M$, the difference $t_m - s_N$ consists of a finite number of the a_k terms which, for large enough $n \geq N$, all appear in $\sum_{k=N+1}^{n} a_k$.

Let $g : \mathbb{N} \to \{0, 1\}$ be the function defined by $g(k) = 1$ if a_k appears in $t_m - s_N$ and $g(k) = 0$ if a_k does not appear in $t_m - s_N$.

Then

$$t_m - s_N = \sum_{k=N+1}^{n} g(k) a_k,$$

and so

$$|t_m - s_N| = \left| \sum_{k=N+1}^{n} g(k) a_k \right| \leq \sum_{k=N+1}^{n} |g(k) a_k| \leq \sum_{k=N+1}^{n} |a_k|.$$

The choice of N_2 guarantees that $|t_m - s_N| < \epsilon/2$ when $m \geq M \geq N_2$, and so

$$\begin{aligned}
|t_m - A| &= |t_m - s_N + s_N - A| \\
&\leq |t_m - s_N| + |s_N - A| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

whenever $m \geq M$, and thus t_m converges to A. $\square$

Appendix: Some Proofs

Proof of Theorem 2.7.1. (i) A partial sum of $\sum_{k=1}^{\infty} ca_k$ is

$$t_m = ca_1 + \cdots + ca_m.$$

A partial sum of $\sum_{k=1}^{\infty} a_k$ is

$$s_m = a_1 + \cdots + a_m.$$

Then $t_m = cs_m$, and since $s_m \to A$, we obtain by the Algebraic Limit Theorem (for sequences) that $t_m \to cA$.

(ii) A partial sum of $\sum_{k=1}^{\infty}(a_k + b_k)$ is

$$w_m = a_1 + b_1 + \cdots + (a_m + b_m) = a_1 + \cdots + a_m + b_1 + \cdots + b_m.$$

Partial sums for $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are

$$s_m = a_1 + \cdots + a_m, \;\; t_m = b_1 + \cdots + b_m.$$

Then $w_m = s_m + t_m$, and since $s_m \to A$ and $t_m \to B$, we have by the Algebraic Limit Theorem for sequences that $w_m \to A + B$. $\square$

Proof of Theorem 2.7.3. For a convergent series, the sequence of partial sums (s_m) is Cauchy by Theorem 2.7.2.

Thus for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|s_n - s_m| < \epsilon$ whenever $n, m \geq N$.

Choosing $n = m + 1$ gives $|s_{m+1} - s_m| < \epsilon$ for all $m \geq N$.

Here $s_{m+1} - s_m = a_{m+1}$, so we have $|a_{m+1}| < \epsilon$ for all $m \geq N$.

This says that $(a_m) \to 0$. $\square$

Proof of Theorem 2.7.4. Both of the comparisons follow immediately from the Cauchy Criterion for Series and the observation that

$$|a_{m+1} + \cdots + a_n| \leq |b_{m+1} + \cdots + b_n|$$

for $n > m$. $\square$

Proof of the convergence of the Geometric Series. A series is called *geometric* if it is of the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \cdots.$$

If $|r| \geq 1$ and $a \neq 0$, this series diverges because the terms do not go to zero.

For $r \neq 1$, the algebraic identity

$$(1 - r)(1 + r + r^2 + \cdots + r^{m-1}) = 1 - r^m$$

enables us to rewrite the partial sum term

$$s_m = a + ar + \cdots + ar^{m-1} = \frac{a(1 - r^m)}{1 - r}.$$

By the Algebraic Limit Theorem we have

$$\lim_{m\to\infty} s_m = \lim_{m\to\infty} \frac{a(1-r^m)}{1-r} = \frac{a}{1-r}\left(1 - \lim_{m\to\infty} r^m\right),$$

where the limit converges to 0 when $|r| < 1$ (which we saw in Lecture 7). Thus, when $|r| < 1$ we conclude that

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$