Math 341 Lecture #13
§2.8: Double Summations and Products of Infinite Series

You may have noticed that there is no homework for this section. No exam will test you on this section.

Recall that we showed for a double indexed array $\{a_{i,j} : i, j \in \mathbb{N}\}$ it could happen that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \neq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

And also conspicuously missing from the Algebraic Limit Theorem for Series was a product rule.

We will address both of these issues here, starting with sums.

One way to define a *partial sum* of $\sum_{i,j=1}^{\infty} a_{ij}$ is by the finite sum

$$s_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$$

for $m, n \in \mathbb{N}$.

Of particular interest here will be the situation when $m = n$, so that the numbers s_{nn} form a sequence of partial sums.

We can apply the theory of sequences to (s_{nn}) to conclude something about the original double sum.

Definition. We say that $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ converges if (s_{nn}) converges, and we write

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \lim_{n\to\infty} s_{nn}.$$

We will see how the theory of absolute convergence overcomes the problem with how we add up a double sum.

Definition. We say the iterated series

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$$

converges if for each fixed $i \in \mathbb{N}$, the series $\sum_{j=1}^{\infty} |a_{ij}|$ converges to some nonnegative real number b_i, and the series $\sum_{i=1}^{\infty} b_i$ converges.

Theorem 2.8.1. If $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$ converges, then both $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ and $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$ converge to the same number, and

$$\lim_{n\to\infty} s_{nn} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

Proof. Suppose that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}| = A \geq 0$.

Then for each fixed $i \in \mathbb{N}$, the series $\sum_{j=1}^{\infty}|a_{ij}|$ converges to a nonnegative real number b_i, and the series $\sum_{i=1}^{\infty} b_i$ converges to A.

The partial sums of $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|a_{ij}|$ are

$$t_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}| \ , m, n \in \mathbb{N}.$$

The set of partial sums $\{t_{mn} : m, n \in \mathbb{N}\}$ is bounded above:

$$t_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}| \leq \sum_{i=1}^{m}\sum_{j=1}^{\infty}|a_{ij}| = \sum_{i=1}^{m} b_i \leq \sum_{i=1}^{\infty} b_i = A.$$

Since the terms $|a_{ij}|$ in the double sum are nonnegative, the sequence (t_{nn}) is increasing.

Then (t_{nn}) is increasing and bounded, and so converges by the Monotone Convergence Theorem.

By the Cauchy Criterion for Sequences, we have that (t_{nn}) is Cauchy: for $\epsilon > 0$ there is $N \in \mathbb{N}$ such that

$$|t_{mm} - t_{nn}| < \epsilon \text{ for all } m, n \geq N.$$

For $m > n \geq N$ we have

$$\begin{aligned}
|s_{mm} - s_{nn}| &= \left|\sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\right| \\
&= \left|\sum_{i=1}^{n}\sum_{j=n+1}^{m} a_{ij} + \sum_{i=n+1}^{m}\sum_{j=1}^{m} a_{ij}\right| \\
&\leq \sum_{i=1}^{n}\sum_{j=n+1}^{m} |a_{ij}| + \sum_{i=n+1}^{m}\sum_{j=1}^{m} |a_{ij}| \\
&= |t_{mm} - t_{nn}| \\
&< \epsilon.
\end{aligned}$$

[To understand this, think about a_{ij} as the entries of a matrix with i the row and j the column, and carefully consider what $s_{mm} - s_{nn}$ means.]

Thus (s_{nn}) is Cauchy, and hence converges to say S.

Now we will show that

$$S = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}.$$

To do this we go back to the bounded set of partial sums t_{mn}, and set

$$B = \sup\{t_{mn} : m, n \in \mathbb{N}\}.$$

Then for $\epsilon > 0$ there is $m_0, n_0 \in \mathbb{N}$ such that

$$B - \frac{\epsilon}{2} < t_{m_0 n_0} \leq B.$$

For $N_1 = \max\{m_0, n_0\}$ we have $t_{m_0 n_0} \leq t_{mn}$ whenever $m, n \geq N_1$ because each $|a_{ij}| \geq 0$. [To see this think again about $|a_{ij}|$ as the (i, j) entry of a matrix, and what N_1 means.]

With $m > n \geq N_1$, we have

$$|s_{mn} - s_{nn}| = \left| \sum_{i=n+1}^{m} \sum_{j=1}^{n} a_{ij} \right| \leq \sum_{i=n+1}^{m} \sum_{j=1}^{n} |a_{ij}| = t_{mn} - t_{nn}.$$

Since $t_{mn} \leq B$ and $B - \epsilon/2 < t_{nn}$ with the latter implying $-t_{nn} < -B + \epsilon/2$, we obtain

$$|s_{mn} - s_{nn}| < B - B + \frac{\epsilon}{2} = \frac{\epsilon}{2}.$$

With $n > m \geq N_1$, a similar argument shows that $|s_{mn} - s_{nn}| < \epsilon/2$.

Now because $(s_{nn}) \to S$, there is $N_2 \in \mathbb{N}$ such that $|s_{nn} - S| < \epsilon/2$ for all $n \geq N$.

With $N_3 = \max\{N_1, N_2\}$ we have for all $m, n \geq N_3$ that

$$|s_{mn} - S| = |s_{mn} - s_{nn} + s_{nn} - S| \leq |s_{mn} - s_{nn}| + |s_{nn} - S| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

For a moment consider $m \geq N_3$ to be fixed and write

$$s_{mn} = \sum_{j=1}^{n} a_{1j} + \sum_{j=1}^{n} a_{2j} + \cdots + \sum_{j=1}^{n} a_{mj}.$$

The hypothesis guarantees that for each fixed $i \in \mathbb{N}$, the series $\sum_{j=1}^{\infty} a_{ij}$ converges (absolutely) to a real number r_i.

By the Algebraic Limit Theorem, we have that

$$\lim_{n\to\infty} s_{mn} = r_1 + r_2 + \cdots + r_m.$$

From $|s_{mn} - S| < \epsilon$ for all $n \geq N_3$, we know that

$$S - \epsilon < s_{mn} < S + \epsilon.$$

By the Order Limit Theorem we obtain for all $m \geq N_3$ that

$$S - \epsilon \leq r_1 + r_2 + \cdots + r_m \leq S + \epsilon.$$

Thus we have for $m \geq N_3$ that

$$|r_1 + r_2 + \cdots + r_m - S| \leq \epsilon$$

which is

$$\left|\sum_{i=1}^{m}\sum_{j=1}^{\infty} a_{ij} - S\right| \leq \epsilon.$$

From this we conclude that

$$\lim_{n\to\infty} s_{nn} = S = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}.$$

To get the other iterated sum equaling S, we only need to show that for each $j \in \mathbb{N}$, the sum $\sum_{i=1}^{\infty} a_{ij}$ converges to some real number c_j; then we argue as we did before to get

$$S = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

Thus both of the iterated sums exist and equal each other. $\square$.

As you might expect now, to have a product rule in the Algebraic Limit Theorem for series, we need to assume that

$$\sum_{i=1}^{\infty} a_i \text{ and } \sum_{j=1}^{\infty} b_j$$

converge absolutely to A and B respectively, and then we can conclude correctly that

$$\left(\sum_{i=1}^{\infty} a_i\right)\left(\sum_{j=1}^{\infty} b_j\right) = AB.$$