## Math 341 Lecture \#14 <br> §3.1, 3.2: The Cantor Set; Open and Closed Sets, Part I

§3.1: The Cantor Set. We are going to construct a "bizarre" nonempty subset of $\mathbb{R}$ through an intersection of nested sets.
We start with the closed interval $C_{0}=[0,1]$.
We form a subset $C_{1}$ by removing the open middle third interval $(1 / 3,2 / 3)$ from $C_{0}$.
In other words,

$$
C_{1}=C_{0}-\left(\frac{1}{3}, \frac{2}{3}\right)=\left[0, \frac{1}{3}\right] \cup\left[\frac{2}{3}, 1\right] .
$$

We form a set $C_{2}$ by removing the open middle third interval from each part of $C_{1}$ :

$$
C_{2}=\left[0, \frac{1}{9}\right] \cup\left[\frac{2}{9}, \frac{1}{3}\right] \cup\left[\frac{2}{3}, \frac{7}{9}\right] \cup\left[\frac{8}{9}, 1\right] .
$$

Continuing this process, we have for each $n=0,1,2,3, \ldots$ a set $C_{n}$ consisting of $2^{n}$ closed intervals each having length $1 / 3^{n}$.
The middle thirds Cantor set is defined by

$$
C=\bigcap_{n=0}^{\infty} C_{n} .
$$

We can think of the Cantor set as what is left of $C_{0}=[0,1]$ after removing the middle third open interval at each step:

$$
C=[0,1]-\left[\left(\frac{1}{3}, \frac{2}{3}\right) \cup\left(\frac{1}{9}, \frac{2}{9}\right) \cup\left(\frac{7}{9}, \frac{8}{9}\right) \cup \cdots\right] .
$$

Can you identify some points that are in $C$ ?
The endpoints 0 and 1 of $C_{0}$ are in all the $C_{n}$ and they are in $C$ too.
How about the endpoints $1 / 3$ and $2 / 3$ of $C_{1}$ ? Yes, these are in all the $C_{n}$ and so in $C$.
In fact for each $k \in \mathbb{N}$, the endpoints of $C_{k}$ are in all the $C_{n}$, and so are in $C$ too.
The endpoints of $C_{n}$ are the form $m / 3^{n}$ for some integer $m$ satisfying $0 \leq m \leq 3^{n}$, and hence are rational.
If $C$ were to just consist of the union of the endpoints of the $C_{n}$, then $C$ would be a subset of $\mathbb{Q}$ and hence $C$ would be countable.
Or could $C$ be uncountable? We shall see.
Another way to probe what is in $C$ is to sum up the lengths of the open intervals that are removed.
An interval of length $1 / 3$ was removed from $C_{0}$ to get $C_{1}$.
Two intervals of length $1 / 9$ were removed from $C_{1}$ to get $C_{2}$.
So in general there were $2^{n-1}$ intervals of length $1 / 3^{n}$ removed from $C_{n-1}$ to get $C_{n}$.

All of the open intervals removed are disjoint, and so we can define the length of $C$ to be 1 minus the total removed:

$$
\begin{aligned}
\frac{1}{3} & +2\left(\frac{1}{9}\right)+4\left(\frac{1}{27}\right)+\cdots+2^{n-1}\left(\frac{1}{3^{n}}\right)+\cdots \\
& =\frac{1}{3}\left[1+\frac{2}{3}+\frac{4}{9}+\cdots+\frac{2^{n-1}}{3^{n-1}}+\cdots\right] \\
& =\frac{1 / 3}{1-2 / 3} \\
& =1
\end{aligned}
$$

[We used the geometric series here to get the sum.]
This says that the length of $C$ is zero! That's weird.
There is another way to assess the "size" of $C$, just to complicate things more.
Magnifying a point (a dimension 0 object) by a factor of 3 results in one copy of the point.

Magnifying a line segment (a dimension 1 object) by a factor of 3 results in 3 copies of the line segment.

Magnifying a square (a dimension 2 object) by a factor of 3 results in 9 copies of the squares.
In each of these cases, the number of copies obtained is $3^{d}$ where $d$ is the dimension of the original object.
Magnifying the Cantor set $C$ by a factor of 3 results in two copies of $C$ : the interval $C_{0}=$ $[0,1]$ becomes $[0,3]$, and removing the open middle third interval results in $[0,1] \cup[2,3]$ in each of which will appear a copy of $C$.
The dimension $d$ of $C$ is the solution of $2=3^{d}$, which gives $d=\ln 2 / \ln 3$ which is not an integer but $0<d<1$.
The point of this is that $C$ has a dimension bigger than that of a point and smaller than that of a line segment.
§3.2: Open and Closed Sets. Recall for $a \in \mathbb{R}$ and $\epsilon>0$, that the $\epsilon$-neighbourhood of $a$ is the set

$$
V_{\epsilon}(a)=\{x \in \mathbb{R}:|x-a|<\epsilon\} .
$$

This is nothing more than the open interval $(a-\epsilon, a+\epsilon)$ centered at $a$ with "radius" $\epsilon$. Definition 3.2.1. A set $O \subseteq \mathbb{R}$ is open if for all points $a \in O$ there exists an $\epsilon>0$ such that $V_{\epsilon}(a) \subseteq O$.

Examples 3.2.2. (a) The set $\mathbb{R}$ is open because for any $a \in \mathbb{R}$ there is $\epsilon>0$ such that $V_{\epsilon}(a) \subseteq \mathbb{R}$.
(b) The empty set $\emptyset$ is open, as there is nothing to check.
(c) Every open interval $(c, d)$ is open because for any $x \in(c, d)$ we take

$$
\epsilon<\min \{x-c, d-x\}
$$

for which we have $V_{\epsilon}(x) \in(c, d)$.
(d) The set $O=(1,2) \cup(3,4)$ is open because in which ever of the two open intervals we choose $x$, we use the method of (c) to show that $V_{\epsilon}(x) \subseteq O$ for some $\epsilon>0$.
(e) The union of any number of open subintervals is open by the logic of (d).

In fact we can say even more.
Theorem 3.2.3. (i) The union of an arbitrary collection of open sets is open. (ii) The intersection of a finite collection of open sets is open.
Proof. (i) Let $\left\{O_{\lambda}: \lambda \in \Lambda\right\}$ be a collection of open sets, and let

$$
O=\bigcup_{\lambda \in \Lambda} O_{\lambda} .
$$

This set has the property that $O_{\lambda} \subseteq O$ for all $\lambda \in \Lambda$.
For an $a \in O$ we need to find a $V_{\epsilon}(a) \subseteq O$.
Because $a \in O$ and $O$ is a union, there is at least one $\lambda^{\prime} \in \Lambda$ such that $a \in O_{\lambda^{\prime}}$.
Since $O_{\lambda^{\prime}}$ is open, there is an $\epsilon>0$ such that $V_{\epsilon}(a) \subseteq O_{\lambda^{\prime}}$.
Since $O_{\lambda^{\prime}} \subseteq O$, we obtain $V_{\epsilon}(a) \subseteq O$, and so $O$ is open.
(ii) Let $\left\{O_{1}, \ldots, O_{N}\right\}$ be a finite collection of open sets, and let

$$
O=\bigcap_{k=1}^{N} O_{k} .
$$

This set has the property that $O \subseteq O_{k}$ for all $k=1, \ldots, N$.
For an $a \in O$ we need to find a $V_{\epsilon}(a) \subseteq O$.
Since $O \subset O_{k}$ for all $k=1, \ldots, N$, we have that $a \in O_{k}$ for all $k=1, \ldots, N$.
Since each $O_{k}$ is open there is an $\epsilon_{k}>0$ such that $V_{\epsilon_{k}}(a) \subseteq O_{k}$.
We want a $V_{\epsilon}(a)$ that is in all of the $O_{k}$, and we do this by finding the smallest value of $\epsilon_{k}$ :

$$
\epsilon=\min \left\{\epsilon_{1}, \ldots, \epsilon_{N}\right\}
$$

Then $V_{\epsilon}(a) \subseteq O_{k}$ for all $k=1, \ldots, N$, and so $V_{\epsilon}(a) \subseteq O$.
Thus $O$ is open.
Any thoughts about why (ii) is not true for the intersection of an infinite collection of open sets?

