Math 341 Lecture #21
§4.3: Continuous Functions

We recall the $\epsilon - \delta$ notion of what it means for a function to be continuous at a point in its domain.

Definition 4.3.1. For a nonempty $A \subseteq \mathbb{R}$, a function $f : A \to \mathbb{R}$ is continuous at a point $c \in A$ if, for all $\epsilon > 0$, there exists $\delta > 0$ such that whenever $|x - c| < \delta$ and $x \in A$, it follows that $|f(x) - f(c)| < \epsilon$.

If f is continuous at every point of A, we say that f is continuous on A.

This definition of continuity at a point looks like a functional limit, except that for continuity we require that $c \in A$, not merely that c is a limit point of A.

We would like to say f is continuous at $c \in A$ by writing

$$\lim_{x \to c} f(x) = f(c).$$

The minor technical difficulty with doing this is when $c \in A$ is an isolated point, but then f is continuous at c in this case because $V_\delta(c) \cap A = \{c\}$ for all small enough $\delta > 0$.

Theorem 4.3.2 (Characterization of Continuity). For a nonempty $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in A$. Then f is continuous at c if and only if any one of the following conditions is met:

(i) for all $\epsilon > 0$ there exists $\delta > 0$ such that $|x - c| < \delta$ and $x \in A$ implies $|f(x) - f(c)| < \epsilon$;

(ii) for all $V_\epsilon(f(c))$ there is $V_\delta(c)$ for which $x \in V_\delta(c) \cap A$ implies $f(x) \in V_\epsilon(f(c))$;

(iii) if $(x_n) \to c$ with $x_n \in A$, then $f(x_n) \to f(c)$.

If c is a limit point of A, then the above conditions are equivalent to

(iv) $\lim_{x \to c} f(x) = f(c)$;

Proof. The equivalence of (i), (iii), and (iv) is simple and straightforward.

We will show the equivalence of (ii) and (iii).

Suppose for all $V_\epsilon(f(c))$ there is $V_\delta(c)$ for which $x \in V_\delta(c) \cap A$ implies $f(x) \in V_\epsilon(f(c))$.

Let (x_n) be a sequence in A converging to c.

Thus there exists $N \in \mathbb{N}$ such that $x_n \in V_\delta(x) \cap A$ for all $n \geq N$.

This implies that $f(x_n) \in V_\epsilon(f(c))$ and so we have $f(x_n) \to f(c)$.

Now, suppose, to the contrary, that there exists ϵ_0 such that for all $V_\delta(c)$ there exists $x \in V_\delta(c) \cap A$ such that $f(x) \notin V_{\epsilon_0}(f(c))$.

For each $\delta_n = 1/n$ we choose $x_n \in V_{\delta_n}(c) \cap A$ for which $f(x_n) \notin V_{\epsilon_0}(f(c))$.

This gives a sequence (x_n) in A which converges to c, but for which $f(x_n) \not\to f(c)$. $\square$

Statement (iii) of this theorem is a new approach to continuity, and hence to discontinuity.

Corollary 4.3.3 (Criterion for Discontinuity). Let $f : A \to \mathbb{R}$, and let $c \in A$ be a limit point of A. If there exists a sequence (x_n) in A with $x_n \to c$ such that $f(x_n) \not\to f(c)$, then f is not continuous at c.

Using the sequential approach to continuity establishes the following.

Theorem 4.3.4 (Algebraic Continuity Theorem). Assume $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are continuous at $c \in A$. Then

(i) $kf(x)$ is continuous at c for all $k \in \mathbb{R}$,
(ii) $f(x) + g(x)$ is continuous at c,
(iii) $f(x)g(x)$ is continuous at c, and
(iv) $f(x)/g(x)$ is continuous at c provided the quotient is defined.

Example 4.3.5. For constants $a, b \in \mathbb{R}$, the function $f(x) = ax + b$ for $x \in A = \mathbb{R}$ is continuous.

To see why, take $c \in A$, and consider

$$|f(x) - (ac + b)| = |ax + b - ac - b| = |a|\ |x - c|.$$

For $\epsilon > 0$ we choose $\delta = \epsilon/|a|$, so that when $x \in V_\delta(c)$ we have

$$|f(x) - (ac + b)| < |a|\epsilon/|a| = \epsilon.$$

By Theorem 4.3.4, it follows that every polynomial is continuous on $\mathbb{R}$, and that every rational function is continuous at those points where the denominator is not zero.

Example 4.3.6. Is the function

$$g(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

continuous at $c = 0$?

For $x \neq 0$, we have

$$|g(x) - g(0)| = |x\sin(1/x) - 0| = |x\sin(1/x)| \leq |x|$$

because $|\sin(1/x)| \leq 1$.

For $\epsilon > 0$ we choose $\delta = \epsilon$ so that when $x \in V_\delta(0)$ we have

$$|g(x) - g(0)| \leq |x| < \delta = \epsilon.$$

Thus $g(x)$ is continuous at $c = 0$.

Example 4.3.8. Show that $f(x) = \sqrt{x}$ is continuous on $A = [0, \infty)$.

For $\epsilon > 0$ and $c \in A$, we need to find $\delta > 0$ such that

$$|f(x) - f(c)| = |\sqrt{x} - \sqrt{c}| < \epsilon$$

when $x \in V_\delta(c) \cap A$.

When $c = 0$ this reduces to $|\sqrt{x} - 0| = \sqrt{x} < \epsilon$.

This gives $x < \epsilon^2$ from which we choose $\delta = \epsilon^2$.

Thus for $x \in V_\delta(0) \cap A$ we have $|\sqrt{x} - 0| = \sqrt{x} < \sqrt{\delta} = \sqrt{\epsilon^2} = \epsilon$, and so we have continuity of $g(x)$ at $c = 0$.

Now for $c > 0$ we are dealing with

$$|\sqrt{x} - \sqrt{c}| = |\sqrt{x} - \sqrt{c}| \, \frac{\sqrt{x} + \sqrt{c}}{\sqrt{x} + \sqrt{c}} = \frac{|x - c|}{\sqrt{x} + \sqrt{c}}.$$

We can control the numerator $|x - c|$ by the choice of δ, so we need to find a way to deal with the denominator $\sqrt{x} + \sqrt{c}$.

Since $\sqrt{x} + \sqrt{c} \geq \sqrt{c} > 0$ we have

$$\frac{1}{\sqrt{x} + \sqrt{c}} \leq \frac{1}{\sqrt{c}}.$$

We then have

$$|\sqrt{x} - \sqrt{c}| \leq \frac{|x - c|}{\sqrt{c}}.$$

The choice of $\delta = \epsilon\sqrt{c}$ then gives us

$$|\sqrt{x} - \sqrt{c}| < \frac{\epsilon\sqrt{c}}{\sqrt{c}} = \epsilon.$$

Thus $f(x) = \sqrt{x}$ is continuous at $c > 0$, and so $f(x)$ is continuous on A.

Recall that the *range* of a function $f : A \to \mathbb{R}$ is the set $f(A) = \{f(x) : x \in A\}$.

Theorem 4.3.9 (Composition of Continuous Functions). Suppose for functions $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ satisfy $f(A) \subseteq B$. If f is continuous at $c \in A$ and g is continuous at $f(c)$, then the composition $g \circ f(x)$ is continuous at c.

The proof of this is a homework problem (4.3.3).