Math 341 Lecture #23
§4.5: The Intermediate Value Theorem

We saw last time for a continuous $f : A \to \mathbb{R}$, that if $K \subseteq A$ is compact, then $f(K)$ is compact; and this led to the Extreme Value Theorem.

There is another topological property of subsets of $\mathbb{R}$ that is preserved by continuous functions, which will lead to the Intermediate Value Theorem.

Theorem 4.5.2 (Preservation of Connectedness). For a continuous function $f : A \to \mathbb{R}$, if $E \subseteq A$ is connected, then $f(E)$ is connected as well.

Proof. Suppose that E is connnected.

The set $f(E)$ is connected if and only if whenever $f(E) = A \cup B$ for disjoint nonempty sets A and B, one of A or B contains a sequence that converges to a point of the other.

Suppose that $f(E) = A \cup B$ for disjoint nonempty sets A and B.

Let

$$C = \{x \in E : f(x) \in A\} = f^{-1}(A)$$

and

$$D = \{x \in E : f(x) \in B\} = f^{-1}(B),$$

which are the preimages of A and B by f respectively.

Since A is a nonempty subset of $f(E)$ the set C is a nonempty subset of E: each $z \in A$ is given by $z = f(x)$ for some $x \in C$.

Similarly D is a nonempty subset of E.

The sets C and D are disjoint because, if there is $x \in C \cap D$, then $f(x) \in A$ and $f(x) \in B$, contradicting the disjointness of A and B.

The sets C and D satisfy $E = C \cup D$ because each $z \in f(E) = A \cup B$ is given by $z = f(x)$ for $x \in C$ or $x \in D$.

The set E is connected, so that one of C or D contains a sequence that converges to a point of the other.

WLOG, suppose C contains a sequence (x_n) that converges to $x \in D$.

The continuity of f implies that $f(x_n) \to f(x)$ where $f(x_n)$ is a sequence in A that converges to $f(x) \in B$.

Therefore $f(E)$ is connected. $\square$

Recall that a subset of $\mathbb{R}$ is connected if and only if it is a (possibly infinite) interval.

We now use this preservation of connectedness by continuous functions to prove the Intermediate Value Theorem.

Theorem 4.5.1 (Intermediate Value Theorem). For $-\infty < a < b < \infty$, if $f : [a, b] \to \mathbb{R}$ is continuous, and if $L \in \mathbb{R}$ satisfies $f(a) < L < f(b)$ or $f(a) > L > f(b)$, then there exists a point $c \in (a, b)$ such that $f(c) = L$.

Proof. The finite length interval $[a, b]$ is connected.

The continuity of f implies by Theorem 4.5.2 that $f([a,b])$ is connected.

Hence the image $f([a,b])$ is an interval.

The values $f(a)$ and $f(b)$ both belong to $f([a,b])$.

When $f(a) < f(b)$, the open interval $(f(a), f(b))$ is then a subset of $f([a,b])$ because the latter is connected.

An L satisfying $f(a) < L < f(b)$ belongs to $(f(a), f(b)) \subseteq f([a,b])$, and so there is $c \in [a,b]$ such that $f(c) = L$.

Because $f(a) < L = f(c) < f(b)$ we have that $c \neq a$ and $c \neq b$, so that $c \in (a,b)$.

A similar argument holds when $f(a) > f(b)$. $\square$.

A typical way the Intermediate Value Theorem is used is to prove the existence of real roots of polynomials.

Example. Does the polynomial

$$p(x) = x^5 - 5x^4 + 11x^3 - 12x^2 + 7x - 1$$

have a real root?

To answer this affirmatively, we find x_1 and x_2 such that $p(x_1) < 0 < p(x_2)$.

The choice of $x_1 = 0$ gives $p(x_1) = -1$, and the choice of $x_2 = 10$ gives $p(x_2) = 59869$.

Because $p(x)$ is continuous, the Intermediate Value Theorem guarantees there is (at least one) $c \in (0, 10)$ such that $p(c) = 0$.

We may not get the exact value of c (there is no analog of the quadratic formula for quintic polynomials).

But we do get a good rational approximation through Newton's method,

$$x_1 = -1,\ x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)} \text{ for } n \in \mathbb{N},$$

which gives $c \approx 0.1999050056$.

Although we do not get the exact value of c, the point here is that this irrational number c exists, and the sequence (x_n) from Newton's methods converges to c.

Corollary (Existence of Roots). For $-\infty < a < b < \infty$, if $f : [a,b] \to \mathbb{R}$ is continuous with $f(a) < 0 < f(b)$, then there exists $c \in (a,b)$ such that $f(c) = 0$.

You will be asked to complete the following proofs of this Corollary using the Axiom of Completeness, and using the Nested Interval Property.

<u>Start of Proof using the Axiom of Completeness.</u> Set

$$K = \{x \in [a,b] : f(x) \leq 0\}.$$

The set K is bounded above by b because $f(b) > 0$, and the set K is nonempty as it contains a because $f(a) < 0$.

Thus $c = \sup K$ exists by the Axiom of Completeness.

The value of $f(c)$ falls into three cases: $f(c) < 0$, $f(c) > 0$, or $f(c) = 0$.

Use the properties of the least upper bound to eliminate the first two cases.

Start of Proof using the Nested Interval Property. Let $I_0 = [a, b]$, and consider this intervals' midpoint $z = (a + b)/2$.

If $f(z) \geq 0$, then choose $a_1 = a$ and $b_1 = z$.

If $f(z) < 0$, then choose $a_1 = z$ and $b_1 = b$.

The interval $I_1 = [a_1, b_1]$ has the property that $f(a_1) < 0$ and $f(b_1) \geq 0$.

Now use induction to construct a sequence of nested closed intervals $I_n = [a_n, b_n]$ such that $f(a_n) < 0$ and $f(b_n) \geq 0$.

Show that $\cap I_n$ consists of one point c for which $a_n \to c$ and $b_n \to c$.

Then use continuity of f to show that $f(c) = 0$.

The Intermediate Value Property. Is the converse of the Intermediate Value Theorem value true? For a function $f : [a, b] \to \mathbb{R}$, does for all $x, y \in [a, b]$ with $x < y$, and for all L between $f(x)$ and $f(y)$, there exist $c \in (x, y)$ such that $f(c) = L$, imply that f is continuous on $[a, b]$?

Example. The function

$$g(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is not continuous at 0.

On the interval $[0, 1]$, for any $x, y \in [0, 1]$ with $x < y$, and any L between $g(x)$ and $g(y)$ there exists $c \in (x, y)$ such that $g(x) = L$.

So the converse of the Intermediate Value Theorem is false as this counterexample proves.

Definition 4.5.3. A function $f : A \to \mathbb{R}$ has the intermediate value property on an interval $[a, b] \subseteq A$ if for all $x, y \in [a, b]$ with $x < y$, and any L between $f(x)$ and $f(y)$, there exists $c \in (x, y)$ such that $f(c) = L$.

Although the converse of the Intermediate Value Theorem is false in general, it is true if we impose monotonicity on the function.

A proof of this is requested in a homework problem (4.5.3).