Math 341 Lecture #24
§4.6: Sets of Discontinuity

We saw at the beginning of Chapter 4 that the set of discontinuities D_f for a function $f : A \to \mathbb{R}$ appeared to always be an F_σ set (a countable union of closed sets).

We will prove this in the case when f is monotone.

Definition. 4.6.1. A function $f : A \to \mathbb{R}$ is *increasing* on A if $f(x) \leq f(y)$ whenever $x < y$ for $x, y \in A$, and is *decreasing* if $f(x) \geq f(y)$ whenever $x < y$ for $x, y \in A$.

A function $f : A \to \mathbb{R}$ is *monotone* if f is either increasing or decreasing.

The function $s(x) = [[x]]$ on $\mathbb{R}$ is monotone increasing.

In showing that $s(x)$ is discontinuous at every integer point, we took a sequence y_n such that $n - 1 < y_n < n$ and $y_n \to n$.

This is a sequence that approaches n from the *left*.

We can talk about functional limits in the same way: from the left or from the right.

Definition. 4.6.2. Given a limit point c of a nonempty set A and a function $f : A \to \mathbb{R}$ we say the limit of $f(x)$ exists from the right and equals L, and write

$$\lim_{x \to c^+} f(x) = L,$$

if for all $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < x - c < \delta$ and $x \in A$.

In terms of sequences this is the same as (x_n) in A with $x_n > c$ and $x_n \to c$, for which $f(x_n) \to L$.

You have it as a homework problem (4.6.3) to state the definition of the limit from the left,

$$\lim_{x \to c^-} f(x) = L.$$

Recall that the limits from the right and from the left are related to the limit.

Theorem 4.6.3. Let $f : A \to \mathbb{R}$ and c a limit point of A. Then $\lim_{x \to c} f(x) = L$ if and only if

$$\lim_{x \to c^-} f(x) = L \text{ and } \lim_{x \to c^+} f(x) = L.$$

The discontinuities of a function can be divided into three categories.

(i) If $\lim_{x \to c} f(x)$ exists but is not equal to $f(c)$, then f has a *removable* discontinuity at c.

(ii) If $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ both exist but are not equal, then f has a *jump* discontinuity at c.

(iii) If $\lim_{x \to c} f(x)$ does not exist for some other reason, then f has an *essential* discontinuity at c.

The third category includes vertical asymptote type discontinuities, like $f(x) = 1/x$ has at $x = 0$, and bounded oscillatory type discontinuities, like $f(x) = \sin(1/x)$ has at $x = 0$.

A monotone function f, though, can have only one type of discontinuity, and this is what makes it easier to identify D_f in this case.

Theorem. If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then

$$\lim_{x \to c^-} f(x) \text{ and } \lim_{x \to c^+} f(x)$$

exist at every at point c in $\mathbb{R}$.

Proof. WLOG, suppose that f is increasing.

For $c \in \mathbb{R}$ consider the nonempty subset $B = \{y = f(x) : x < c\}$ of $\mathbb{R}$.

Since f is increasing, the number $f(c)$ is an upper bound for A.

By the Axiom of Completeness, the number $\sup B$ exists.

The claim is that

$$\lim_{x \to c^-} f(x) = \sup B.$$

For $L = \sup B$, we have that for all $\epsilon > 0$ there exist $y_\epsilon \in B$ such that $L - \epsilon < y_\epsilon \leq L$.

Since $y_\epsilon \in B$, there is $x_\epsilon < c$ such that $f(x_\epsilon) = y_\epsilon$.

For any sequence (x_n) with $x_n < c$ and $x_n \to c$, there exists $N \in \mathbb{N}$ such that $x_\epsilon \leq x_n < c$ for all $n \geq N$.

Thus using the monotonicity of f, we have

$$L - \epsilon < y_\epsilon = f(x_\epsilon) \leq f(x_n) \leq L < L + \epsilon \text{ for all } n \geq N.$$

This says that $f(x_n) \to L$, and so $\lim_{x \to c^-} f(x)$ exists.

In a similar manner we show that $\lim_{x \to c^+} f(x)$ exists. $\square$

Corollary (Exercise 4.6.5). A monotone function $f : \mathbb{R} \to \mathbb{R}$ can have only jump discontinuities.

Proof. By the Theorem, we have for each $c \in \mathbb{R}$ that

$$\lim_{x \to c^-} f(x), \ \lim_{x \to c^+} f(x)$$

both exist.

When these two limits agree, the function f is continuous at c by Theorem 4.6.3.

When these two limits disagree, the function f has a jump discontinuity with a jump of

$$\lim_{x \to c^+} f(x) - \lim_{x \to c^-} f(x)$$

at c.

The only discontinuities that a monotone function can have are jump discontinuities. $\square$

Recall that the monotone function $s(x) = [[x]]$ on $\mathbb{R}$ has $D_s = \mathbb{Z}$, i.e., a countable set of points where $s(x)$ is not continuous.

You have it as a homework problem (4.6.6) to show for a monotone function f that there exists a bijection between D_f and a subset of $\mathbb{Q}$.

Since every subset of $\mathbb{Q}$ is an F_σ set, we will have shown that D_f is an F_σ set when f is monotone.