Math 341 Lecture #26
§5.3: The Mean Value Theorem, Part I

The simple "observation" for a differentiable function $f : [a, b] \to \mathbb{R}$ that there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

is known as the Mean Value Theorem, and is the cornerstone of almost every major theorem about differentiation there is (as we shall see).

Theorem 5.3.1 (Rolle's Theorem). Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, differentiable on (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof. The continuity of f on the compact set $[a, b]$ implies that f attains a maximum value and a minimum value.

If the maximum value occurs at one endpoint of $[a, b]$, and the minimum value occurs at the other endpoint of $[a, b]$, then f is a constant function, so that $f'(x) = 0$ for all $x \in (a, b)$, and we can choose any $c \in (a, b)$.

If the maximum value or the minimum value occurs at an interior point c of $[a, b]$, then by the Interior Extremum Theorem we have $f'(c) = 0$. □

Theorem 5.3.2 (Mean Value Theorem). If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. We reduce the general case of the Mean Value Theorem to the special case of Rolle's Theorem by constructing a new function from f.

Define a function $g : [a, b] \to \mathbb{R}$ by

$$g(x) = f(x) - \left[\left(\frac{f(b) - f(a)}{b - a}\right)(x - a) + f(a)\right].$$

This function $g(x)$ measures the vertical distance from the graph of $f(x)$ to the graph of the line connecting the points $(a, f(a))$ and $(b, f(b))$.

One readily verifies that g is continuous on $[a, b]$ (because f is), and that g is differentiable on (a, b) (because f is).

We evaluate g at the endpoints:

$$g(a) = f(a) - [0 + f(a)] = 0, \quad g(b) = f(b) - [f(b) - f(a) + f(a)] = 0.$$

We can now applies Rolle's Theorem to obtain the existence of $c \in (a, b)$ such that $g'(c) = 0$.

We then translate this back to f:

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Rearranging this gives

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

which is the desired result. $\square$

We know for a constant function $f(x) = k$ on an interval A that $f'(x) = 0$ for all $x \in A$.

But how do we prove the converse? With the Mean Value Theorem.

Corollary 5.3.3. For an interval A, if $g : A \to \mathbb{R}$ is differentiable and satisfies $g'(x) = 0$ for all $x \in A$, then $g(x) = k$ for some constant $k \in \mathbb{R}$.

Proof. Take $x, y \in A$ and WLOG suppose $x < y$.

Applying the Mean Value Theorem to g on $[x, y]$ there exists $c \in (x, y)$ such that

$$g'(c) = \frac{g(y) - g(x)}{y - x}.$$

By hypothesis, we have that $g'(c) = 0$, so that as $x \neq y$, we have $g(x) = g(y)$.

Set k equal to this common value.

The arbitrariness of x and y now implies that $g(x) = k$ for all $x \in A$. $\square$.

Another consequence of the Mean Value Theorem is the familiar result that two antiderivatives of a continuous function differ by a constant.

Corollary 5.3.4. If f and g are differentiable on an interval A and satisfy $f'(x) = g'(x)$ for all $x \in A$, then $f(x) = g(x) + k$ for some constant $k \in \mathbb{R}$.

Proof. The function $h(x) = f(x) - g(x)$ is differentiable on A and satisfies $h'(x) = 0$ for all $x \in A$.

By Corollary 5.3.3., we know that $h(x) = k$ for some constant $k \in \mathbb{R}$, so that $f(x) = g(x) + k$. $\square$

Example. Let f be a differentiable function on $[0, 3]$ where $f(0) = 1$, $f(1) = 3$, $f(2) = 1$, and $f(3) = 2$, and $f'(x) \geq 1$ for $x \in (0, 1)$.

Now there will be lots of differentiable functions whose graphs pass through the points $(0, 1)$, $(1, 3)$, $(2, 1)$, and $(3, 2)$, and have $f'(x) \geq 1$ for all $x \in (0, 1)$.

What properties do all of these differentiable functions have?

Since $f'(x) \geq 1$ for all $x \in (0, 1)$, we know that f is increasing on $(0, 1)$.

The function $g(x) = x - f(x)$ is continuous on $[0, 3]$, and since $g(0) = -1$ and $g(3) = 3 - 2 = 1$, there exists by the Intermediate Value Theorem a point $d \in (0, 3)$ such that $g(d) = 0$, or $f(d) = d$.

That is, the graph of f crosses the graph of $y = x$ at $x = d$.

For any subinterval $[a, b]$ of $[0, 3]$, the function f is continuous on $[a, b]$ and differentiable on (a, b).

By the Mean Value Theorem there exists $c_1 \in (1, 2)$ such that

$$f'(c_1) = \frac{f(2) - f(1)}{2 - 1} = \frac{1 - 3}{1} = -2.$$

By the Mean Value Theorem there exists $c_2 \in (2, 3)$ such that

$$f'(c_2) = \frac{f(3) - f(2)}{3 - 2} = \frac{2 - 1}{3 - 2} = 1.$$

Since f is differentible on $[c_1, c_2]$, there is by Darboux's Theorem a point $c_3 \in (c_1, c_2) \subseteq (1, 3)$ such that $f'(c_3) = 0$.

The following result is a generalization of the Mean Value Theorem due to Cauchy, and is key to proving L'Hospital's Rule.

Theorem 5.3.5. If f and g are continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

If $g'(x) \neq 0$ for all $x \in (a, b)$, then we have

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - b(a)}.$$

Proof. The function

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$$

is continuous on $[a, b]$ and differentiable on (a, b).

Since

$$\begin{aligned} h(b) &= [f(b) - f(a)]g(b) - [g(b) - g(a)]f(b) \\ &= f(b)g(b) - f(a)g(b) - g(b)f(b) + g(a)g(b) \\ &= -f(a)g(b) + g(a)f(b), \\ h(a) &= [f(b) - f(a)]g(a) - [g(b) - g(a)]f(a) \\ &= f(b)g(a) - f(a)g(a) - g(b)f(a) + g(a)f(a) \\ &= f(b)g(a) - g(b)f(a), \end{aligned}$$

we have that $h(b) = h(a)$.

Applying Rolle's Theorem gives the existence of $c \in (a, b)$ such that $h'(c) = 0$, which unwrapped gives the desired conclusion. □