Math 341 Lecture #27
§5.3: The Mean Value Theorem, Part II
§5.4: A Continuous Nowhere Differentiable Function

Theorem 5.3.6 (L'Hospital's Rule: 0/0 case). Assume f and g are continuous functions defined on an interval containing a, and assume that f and g are differentiable on this interval (with the possible exception of a). If $f(a) = 0$ and $g(a) = 0$, then

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L \ \Rightarrow\ \lim_{x\to a} \frac{f(x)}{g(x)} = L.$$

Proof. We get to start with assuming that

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L.$$

Then for any $\epsilon > 0$ there exists $\delta > 0$ such that whenever $0 < |x - a| < \delta$ we have

$$\left|\frac{f'(x)}{g'(x)} - L\right| < \epsilon.$$

This says that $f'(x)/g'(x)$ exists when $0 < |x - a| < \delta$, so that $g'(x) \neq 0$ when $0 < |x - a| < \delta$.

By the Generalized Mean Value Theorem, with $0 < h < \delta$, we have the existence of $c \in (a, a + h)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(a + h) - f(a)}{g(a + h) - g(a)}.$$

By hypothesis, we know that $f(a) = 0$ and $g(a) = 0$, so that

$$\frac{f'(c)}{g'(c)} = \frac{f(a + h)}{g(a + h)}.$$

Since $0 < c - a < a + h - a < h < \delta$, we have

$$\left|\frac{f(a + h)}{g(a + h)} - L\right| = \left|\frac{f'(c)}{g'(c)} - L\right| < \epsilon.$$

For $x = a + h > a$, we have $0 < x - a = h < \delta$, so that

$$\left|\frac{f(x)}{g(x)} - L\right| < \epsilon.$$

This says that

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = L.$$

In a similar manner we obtain the other one-sided limit,

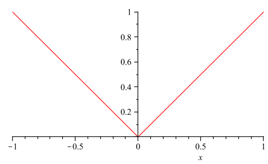
$$\lim_{x\to a^-} \frac{f(x)}{g(x)} = L,$$

thus obtaining the desired result. $\square$

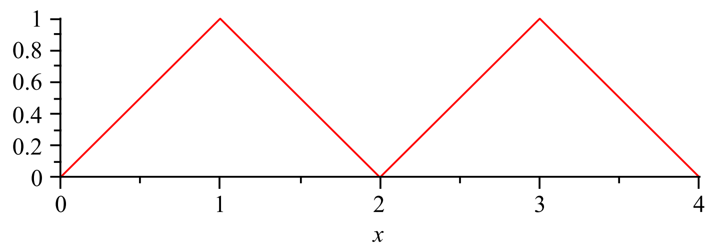
A Continuous Nowhere-Differentiable Function. We are going to construct a convergent series of functions whose limit function is a continuous nowhere differentiable function on $\mathbb{R}$.

This may seems a pathological example, but it is more the norm than you think.

We start with the function $h(x) = |x|$ on $[-1, 1]$. Here is the graph of $h(x)$ on $[-1, 1]$.

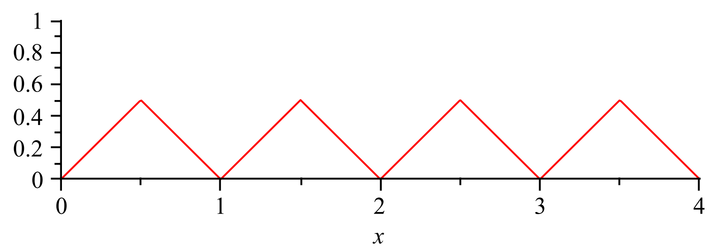


We extend this $h(x)$ periodically to a function $h_0(x)$ on $\mathbb{R}$ defined by $h(x+2) = h(x)$. Here is the graph of the $h_0(x)$ on $[0, 4]$.

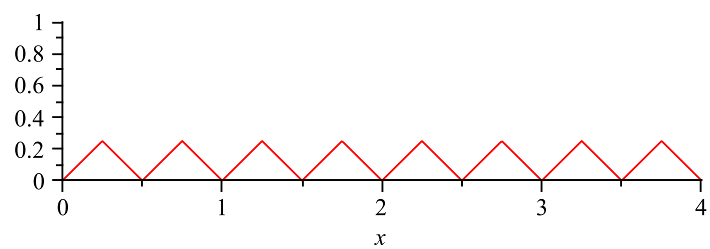


The periodic function $h_0(x)$ is even and satisfies $|h_0(x)| \leq 1$ for all $x \in \mathbb{R}$.

What does the graph of $h_1(x) = (1/2)h_0(2x)$ look like? Here is this graph on $[0, 4]$.



What does the graph of $h_2(x) = (1/4)h_0(4x)$ look like? Here is this graph on $[0, 4]$.



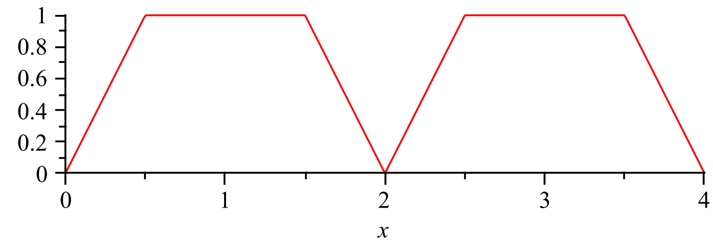
From these graphs, we see that for each $n \in \mathbb{N}$, the even function

$$h_n(x) = (1/2^n)h_0(2^n x)$$

is continuous on $\mathbb{R}$ whose graph consists of line segments of alternating slopes, and is not differentiable at points of the form $p/2^n$ for integers p.

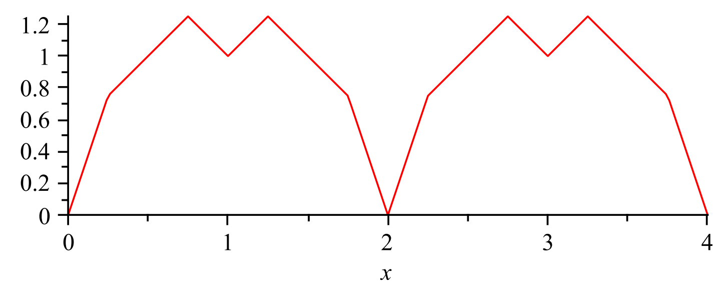
We are going to add the functions $h_0(x)$, $h_1(x)$, $h_2(x)$, etc.

Here is the graph of $h_0(x) + h_1(x)$ on $[0, 4]$.



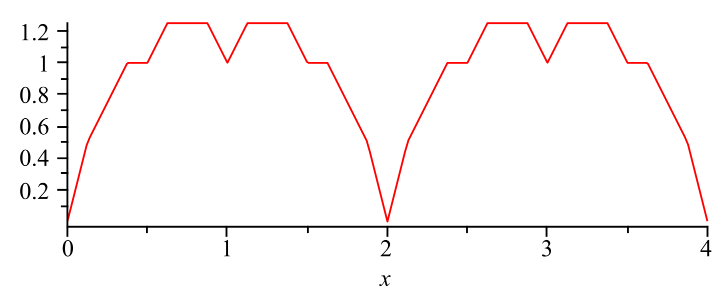
Do you see where this function is not differentiable? At the points 1/2, 3/2, 2, 5/2, 7/2, etc.

Here is the graph of $h_0(x) + h_1(x) + h_2(x)$ on $[0, 4]$.



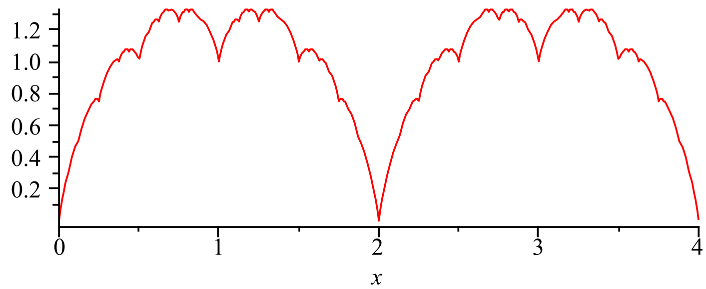
Do you see where this functions is not differentiable? At the points 1/4, 3/4, 1, 5/4, 7/4, 2, etc.

Here is the graph of $\sum_{n=0}^{3} h_n(x)$ on $[0, 4]$.



Do you see where the function is not differentiable? At points of the form $p/8 = p/2^3$ for integers p.

Here is the graph of $\sum_{n=1}^{7} h_n(x)$ on $[0, 4]$.



This function is not differentiable at points of the points $p/2^7$ for integers p.

The claim is that the infinite sum

$$g(x) = \sum_{n=0}^{\infty} h_n(x) = \sum_{n=0}^{\infty} (1/2^n) h_0(2^n x)$$

is a continuous nowhere differentiable even function on $\mathbb{R}$.

First thing to settle is that this infinite series actually defines a function on $\mathbb{R}$.

Recall that $h_0(x)$ satisfies $|h_0(x)| \leq 1$ for all $x \in \mathbb{R}$.

Thus $h_0(2^n x)$ also satisfies $|h_0(2^n x)| \leq 1$ for all $x \in \mathbb{R}$ and all $n = 0, 1, 2, 3, \ldots$.

So for each $x \in \mathbb{R}$ we have

$$\sum_{n=0}^{\infty} \left| \frac{h_0(2^n x)}{2^n} \right| \leq \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1 - 1/2} = 2$$

by the geometric series.

It follows by the Absolute Convergence Test (Theorem 2.7.6) that $g(x)$ is a convergent series for each $x \in \mathbb{R}$, and so $g(x)$ is a properly defined function on $\mathbb{R}$.

Remember that this says that for each $x \in \mathbb{R}$, the sequence of partial sums

$$g_m(x) = \sum_{n=0}^{m} h_n(x)$$

converges to $g(x)$ as $m \to \infty$.

Because each function h_n is continuous on $\mathbb{R}$, it follows for each $m = 0, 1, 2, 3, \ldots$, that the function $g_m(x)$ is continuous on $\mathbb{R}$.

We will see later, in Chapter 6, that $g(x)$ is continuous.

Appendix. We will deal here with the harder problem of showing that g is nowhere differentiable function.

Looking at the graph of $g_7(x)$ above, it appears that g is not differentiable at any integer point c.

For simplicity we take $c = 0$, and consider the sequence $x_m = 1/2^m$ for $m = 0, 1, 2, 3, \ldots$.

In the rise over run formulation of the derivative, we will need the value of $g(0)$.

Since $h_n(x) = (1/2^n)h_0(2^n x)$, we have that $h_n(0) = 0$ for all $n = 0, 1, 2, 3, \ldots$, and so $g(0) = 0$ as well.

We will find the value of

$$\frac{g(x_m) - g(0)}{x_m - 0} = 2^m g(x_m).$$

To do this we compute

$$g(x_m) = \sum_{n=0}^{\infty} h_n(x_m) = \sum_{n=0}^{\infty} \frac{h_0(2^n x_m)}{2^n} = \sum_{n=0}^{\infty} \frac{h_0(2^n/2^m)}{2^n}.$$

When $n \geq m+1$, then $2^n/2^m = 2^{n-m}$ is an even integer, and so $h_0(2^n/2^m) = 0$ for all $n \geq m+1$, hence that

$$\sum_{n=0}^{\infty} \frac{h_0(2^n/2^m)}{2^n} = \sum_{n=0}^{m} \frac{h_0(2^n/2^m)}{2^n}.$$

For $0 \leq n \leq m$ we have that $h_0(2^n/2^m) = h_0(2^{n-m}) = 2^{n-m}$, so that

$$\sum_{n=0}^{m} \frac{h_0(2^n/2^m)}{2^n} = \sum_{n=0}^{m} \frac{2^{n-m}}{2^n} = \sum_{n=0}^{m} \frac{1}{2^m} = \frac{(m+1)}{2^m}.$$

We then have that

$$2^m g(x_m) = 2^m \left(\frac{m+1}{2^m} \right) = m+1.$$

Thus

$$\lim_{m\to\infty} \frac{g(x_m) - g(0)}{x_m - 0} = \lim_{m\to\infty} (m+1) = \infty,$$

and so g is not differentiable at $c = 0$.

In a similar way the sequence $x_m = -1/2^m$ yields

$$\lim_{m\to\infty} \frac{g(x_m) - g(0)}{x_m - 0} = -\infty.$$

This shows that $g(x)$ has a "cusp" at $c = 0$.

In a similar manner, one can show that g is not differentiable at $c = 1$, $c = 1/2$, and at $c = p/2^k$ for $p \in \mathbb{Z}$ and $k \in \mathbb{N} \cup \{0\}$.

Points of the form $p/2^k$ for $p \in \mathbb{Z}$ and $k \in \mathbb{N} \cup \{0\}$ are called *dyadic* points.

The dyadic points form a dense subset of $\mathbb{R}$, and so we then have that $g(x)$ is not differentiable on this dense subset.

Now assume that x is not a dyadic point.

Then for each $m \in \mathbb{N} \cup \{0\}$, the point x falls between two adjacent dyadic points,

$$\frac{p}{2^m} < x < \frac{p+1}{2^m}$$

for some $p \in \mathbb{Z}$.

For the sequences $x_m = p/2^m$ and $y_m = (p+1)/2^m$ we have $x_m < x < y_m$ and $x_m \to x$ and $y_m \to x$.

One can show for each m that g_m is differentiable at each non-dyadic x, and the sequence of derivatives $(g_m'(x))$ satsifies

$$|g_{m+1}'(x) - g_m'(x)| = 1, \; m \in \mathbb{N} \cup \{0\}.$$

Specifically the sequence $(g_m'(x))$ eventually alternates between -1 and 1.

If one can show that the inequalities

$$\frac{g(y_m) - g(x)}{y_m - x} \leq g_m'(x) \leq \frac{g(x_m) - g(x)}{x_m - x}$$

hold for all m, then it follows that when

$$\lim_{m\to\infty} \frac{g(y_m) - g(x)}{y_m - x}, \quad \lim_{m\to\infty} \frac{g(x_m) - g(x)}{x_m - x}$$

both exist they can not have same value because the sequence $(g_m'(x))$ eventually alternates between -1 and 1, keeping the two rise over runs apart from each other.

Now we show that the inequalities hold for all $m = 0, 1, 2, 3, \dots$.

Recalling that $h_0(x) = 0$ whenever x is an even integer, we have that

$$\begin{aligned} g(x_m) &= \sum_{n=0}^{\infty} \frac{h_0(2^n x_m)}{2^n} \\ &= \sum_{n=0}^{\infty} \frac{h_0(p2^{n-m})}{2^n} \\ &= \sum_{n=0}^{m} \frac{h_0(p2^{n-m})}{2^n} \\ &= g_m(x_m). \end{aligned}$$

For x, the sequence $g_m(x)$ is increasing because $h_0 \geq 0$, so that $g_m(x) \leq g(x)$ and hence $-g(x) \leq -g_m(x)$.

Thus we have that

$$g(x_m) - g(x) = g_m(x_m) - g(x) \leq g_m(x_m) - g_m(x).$$

Since $x_m < x$, then $x_m - x < 0$, so that

$$\frac{g(x_m) - g(x)}{x_m - x} \geq \frac{g_m(x_m) - g_m(x)}{x_m - x}.$$

The graph of the function g_m on the interval $[x_m, y_m]$ is a straight line segment (in fact, g_m is a continuous piecewise linear function) so that

$$\frac{g_m(x_m) - g_m(x)}{x_m - x} = g_m'(x).$$

Thus we have that

$$g_m'(x) \leq \frac{g(x_m) - g(x)}{x_m - x}.$$

In a similar way we get

$$\frac{g(y_m) - g(x)}{y_m - x} \leq g_m'(x).$$

Thus g is not differentiable at the non-dyadic x, and so g is nowhere differentiable.