# Math 341 Lecture #28
## §6.2: Uniform Convergence

Recall that we constructed a continuous nowhere differentiable function by way of a convergent series of continuous functions:

$$g(x) = \sum_{n=0}^{\infty} h_n(x).$$

We showed that for each $x$ the series converged, that the sequence of partials sums converged.

We give this "type" of convergence a name.

**Definition 6.2.1.** For each $n \in \mathbb{N}$ let $f_n : A \to \mathbb{R}$ for $A \subseteq \mathbb{R}$. The sequence $(f_n)$ *converges pointwise on* $A$ to a function $f : A \to \mathbb{R}$ if for all $x \in A$ the sequence of real numbers $f_n(x)$ converges to $f(x)$.

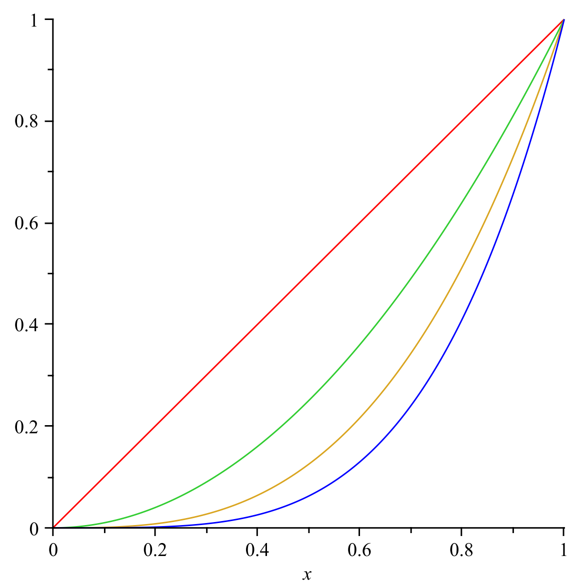
Notations for this pointwise convergence of $f_n$ to $f$ on $A$ are

$$f_n \to f, \quad \lim f_n = f, \quad \lim_{n\to\infty} f_n(x) = f(x),$$

where the domain $A$, not written, is understood.

**Example 6.2.2.** (ii) For $n \in \mathbb{N}$, let $g_n(x) = x^n$ on the domain $A = [0, 1]$.

Here are the graphs of $g_1$, $g_2$, $g_3$, $g_4$.




We notice that $g_n(1) = 1$ for all $n \in \mathbb{N}$, so that

$$\lim_{n\to\infty} g_n(1) = 1.$$

For $0 \le x < 1$, we have that

$$\lim_{n\to\infty} g_n(x) = 0.$$

Although each $g_n$ is differentiable (and hence continuous) on $A$, the limit function

$$g(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1, \end{cases}$$

is not continuous on $A$.

We have $g_n \to g$ pointwise: this sequence of continuous functions converges to a discontinuous function.

This says that pointwise convergence of a series of continuous functions is not enough to guarantee the limit function if continuous; we need a stronger "type" of convergence.

Definition 6.2.3. Let $f_n$ be a sequence of functions defined on $A \subseteq \mathbb{R}$. We say that $(f_n)$ *converges uniformly on* $A$ to a limit function $f$ on $A$ if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for all $x \in A$ whenever $n \geq N$.

Certainly, uniform convergence implies pointwise convergence, but the converse is false (as we have seen), so that uniform convergence is a stronger "type" of convergence than pointwise convergence.

Example. 6.2.4. (i) Let

$$g_n(x) = \frac{1}{n(1+x^2)}.$$

For any $x \in \mathbb{R}$, we see that $g_n(x) \to 0$, so that $g_n$ converges pointwise to $g(x) = 0$.

Is this convergence uniform on $\mathbb{R}$?

Well, since $1/(x^2+1) \leq 1$ for all $x \in \mathbb{R}$, we have

$$|g_n(x) - g(x)| = \left| \frac{1}{n(x^2+1)} - 0 \right| \leq \frac{1}{n}.$$

Thus for a given $\epsilon > 0$ we can choose $N \geq 1/\epsilon$, a choice of $N$ that is independent of $x \in \mathbb{R}$, so that $|g_n(x) - g(x)| < \epsilon$ for all $x \in \mathbb{R}$ whenever $n \geq N$.

This says that $g_n$ converges uniformly to $g$ on $\mathbb{R}$.

(ii) It is "easy" to guess that the sequence of functions

$$f_n(x) = \frac{x^2 + nx}{n}$$

converges pointwise to $f(x) = x$ on all of $\mathbb{R}$.

Investigating whether this convergence is uniform, we consider

$$|f_n(x) - f(x)| = \left| \frac{x^2 + nx}{n} - x \right| = \left| \frac{x^2 + nx - nx}{n} \right| = \frac{x^2}{n}.$$

Asking that $|f_n(x) - f(x)| < \epsilon$ for $\epsilon > 0$ requires that we choose $N \in \mathbb{N}$ according to

$$N > \frac{x^2}{\epsilon}.$$

This says that we cannot choose one value of $N$ that works for all $x \in \mathbb{R}$, and so the pointwise convergence of $f_n$ to $f$ is not uniform on $\mathbb{R}$.

If instead, we restrict the domain of $f_n$ to the compact subset $[-b, b]$ (for $b > 0$), then we can get uniform convergence of $f_n$ to $f$ by choosing

$$N > \frac{b^2}{\epsilon}.$$

That is we have for all $x \in [-b, b]$ that

$$|f_n(x) - f(x)| = \frac{x^2}{n} \leq \frac{b^2}{n} \leq \frac{b^2}{N} < \epsilon$$

for all $n \geq N$.

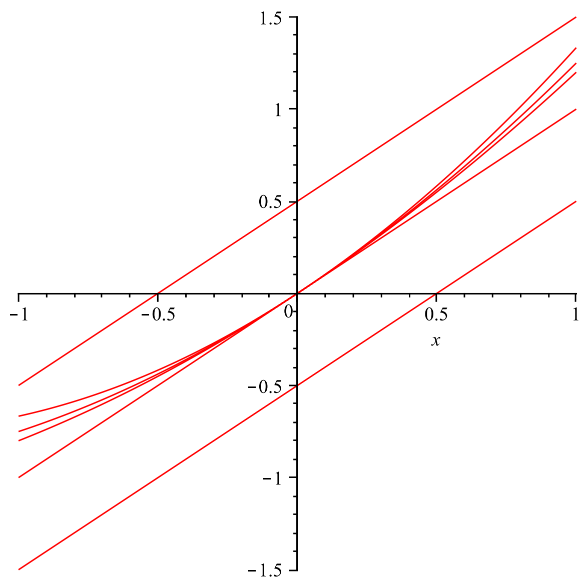
For $b = 1$, what does uniform convergence of $f_n$ to $f$ on $[-1, 1]$ means geometrically?

For $\epsilon = 1/2$ and $b = 1$ we have that $N = 3 > 1^2/(1/2) = 2$, and we get

$$|f_n(x) - f(x)| < 1/2 \text{ or } f(x) - 1/2 < f_n(x) < f(x) + 1/2$$

for all $x \in [-1, 1]$ and all $n \geq 3$.

Here are the graphs of $f(x) = x$, $f(x) + 1/2$, $f(x) - 1/2$, and $f_n(x)$ for $n = 3, 4, 5$.




What do you notice about the graphs of $f_n(x)$ for $n = 3, 4, 5$? Each lies within the $\epsilon = 1/2$ "tube" about $f(x)$ on $[-1, 1]$.

This is what uniform convergence means geometrically.

Recall the Cauchy Criterion for the convergence of a sequence of real numbers did not require a "guess" for the limit.

We have a similar criterion for uniform convergence.

Theorem 6.2.5 (Cauchy Criterion for Uniform Convergence). A sequence of functions $(f_n)$ defined on $A \subseteq \mathbb{R}$ converges uniformly on $A$ if and only if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$|f_n(x) - f_m(x)| < \epsilon$$

for all $n, m \geq N$ and all $x \in A$.

Proof. Suppose that $f_n$ converges uniformly to $f$ on $A$.

Then for $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon/2$ for all $n \geq N$ and all $x \in A$.

Thus for $n, m \geq N$ and any $x \in A$ we have

$$\begin{aligned}
|f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\
&\leq |f_n(x) - f(x)| + |f_m(x) - f(x)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Now suppose that for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$ for all $n, m \geq N$ and all $x \in A$.

Then for each $x \in A$, the sequence $(f_n(x))$ is Cauchy sequence of real numbers, and therefore it converges to a real number, call it $f(x)$.

We have found a function $f : A \to \mathbb{R}$ which is the pointwise limit of $f_n$.

By the Algebraic Limit and Order Limit Theorems we have for all $n \geq N$ and all $x \in A$ that

$$|f_n(x) - f(x)| = \lim_{m \to \infty} |f_n(x) - f_m(x)| \leq \epsilon$$

which says that $f_n$ converges uniformly to $f$ on $A$. $\square$

The stronger assumption of uniform convergence is enough to guarantee that the limit function of a sequence of continuous functions is continuous.

Theorem 6.2.6 (Continuous Limit Theorem). Let $(f_n)$ be a sequence of functions defined on $A \subseteq \mathbb{R}$ that converges uniformly on $A$ to $f$. If each $f_n$ is continuous at $c \in A$, then $f$ is continuous at $c$ too.

Proof. Fix $c \in A$, and for $\epsilon > 0$ choose $N \in \mathbb{N}$ such that for all $x \in A$ we have

$$|f_N(x) - f(x)| < \frac{\epsilon}{3}.$$

By the continuity of $f_N$ at $c$ there exists $\delta > 0$ such that whenever $|x - c| < \delta$ we have

$$|f_N(x) - f_N(c)| < \frac{\epsilon}{3}.$$

Thus

$$|f(x) - f(c)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

whenever $|x - c| < \delta$, and so $f$ is continuous at $c$. $\square$