Math 341 Lecture #29
§6.3: Uniform Convergence and Differentiation

We have seen that a pointwise converging sequence of continuous functions need not have a continuous limit function; we needed uniform convergence to get continuity of the limit function.

What can we say about the differentiability of the limit function of a pointwise converging sequence of differentiable functions?

The sequence of differentiable $h_n(x) = x^{1+1/(2n-1)}$, $x \in [-1, 1]$, converges pointwise to the nondifferentiable $h(x) = x$; we will need to assume more about the pointwise converging sequence of differentiable functions to ensure that the limit function is differentiable.

Theorem 6.3.1 (Differentiable Limit Theorem). Let $f_n \to f$ pointwise on the closed interval $[a, b]$, and assume that each f_n is differentiable. If (f_n') converges uniformly on $[a, b]$ to a function g, then f is differentiable and $f' = g$.

Proof. Let $\epsilon > 0$ and fix $c \in [a, b]$.

Our goal is to show that $f'(c)$ exists and equals $g(c)$.

To this end, we will show the existence of $\delta > 0$ such that for all $0 < |x - c| < \delta$, with $x \in [a, b]$, we have

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| < \epsilon$$

which implies that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists and is equal to $g(c)$.

The way forward is to replace $(f(x) - f(c))/(x - c) - g(c)$ with expressions we can hopefully control:

$$\begin{aligned}
\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| &= \left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} + \frac{f_n(x) - f_n(c)}{x - c} \right. \\
&\qquad \left. - f_n'(c) + f_n'(c) - g(c) \right| \\
&\le \left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| \\
&\qquad + \left| \frac{f_n(x) - f_n(c)}{x - c} - f_n'(c) \right| + |f_n'(c) - g(c)|.
\end{aligned}$$

The second and third expressions we can control respectively by the differentiability of f_n and the uniformly convergence of f_n' to g.

It is the first term that will take some effort to control.

Let $x \in [a, b]$ such that $x \neq c$.

Then either $x > c$ or $x < c$; assume that $x > c$ (the other case is similar).

For $m, n \in \mathbb{N}$, the function $f_m - f_n$ is differentiable on $[c, x]$, and so by the Mean Value Theorem there exists $\alpha \in (c, x)$ such that

$$f'_m(\alpha) - f'_n(\alpha) = \frac{(f_m(x) - f_n(x)) - (f_m(c) - f_n(c))}{x - c}.$$

Rearranging the right hand side of this gives

$$f'_m(\alpha) - f'_n(\alpha) = \frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c}.$$

Since (f'_n) converges uniformly to g we have by the Cauchy Criterion for Uniform Convergence the existence of $N_1 \in \mathbb{N}$ such that for all $m, n \geq N_1$ we have

$$|f'_m(\alpha) - f'_n(\alpha)| < \frac{\epsilon}{3}.$$

Without the uniform convergence of f'_n to g, we would not have been able to do this because the α depends on m, n.

Putting the pieces together we have

$$\left| \frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| = |f'_m(\alpha) - f'_n(\alpha)| < \frac{\epsilon}{3}.$$

Because $f_n \to f$ (pointwise convergence), we can use the Order Limit Theorem to control the first expression by taking the limit as $m \to \infty$:

$$\left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| \leq \frac{\epsilon}{3}.$$

By the uniform convergence of f'_m to g we have control of the third expression: there is $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$ we have

$$|f'_n(c) - g(c)| < \frac{\epsilon}{3}.$$

We settle the choice of N needed to control the second expression: $N = \max\{N_1, N_2\}$.

Then the differentiability of f_N gives the existence of $\delta > 0$ such that when $0 < |x - c| < \delta$ with $x \in [a, b]$, we have

$$\left| \frac{f_N(x) - f_N(c)}{x - c} - f'_N(c) \right| < \frac{\epsilon}{3}.$$

We now have the choice of N and $0 < |x - c| < \delta$ with $x \in [a, b]$, such that

$$\begin{aligned} \left| \frac{f(x) - f(c)}{x - c} - g(c) \right| &\leq \left| \frac{f(x) - f(c)}{x - c} - \frac{f_N(x) - f_N(c)}{x - c} \right| + \left| \frac{f_N(x) - f_N(c)}{x - c} - f'_N(c) \right| \\ &\quad + |f'_N(c) - g(c)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

This shows that $f'(c)$ exists and is equal to $g(c)$.

As $c \in [a, b]$ was arbitrary, we conclude that f is differentiable on $[a, b]$ and satisfies $f' = g$. $\square$

In the hypothesis of Theorem 6.3.1 we assumed that $f_n \to f$ and (f_n') converged uniformly to g.

But doesn't uniform convergence of f_n' to g almost, nearly, within ϵ, imply the uniform convergence of f_n to f? Not quite as it is possible for f_n to diverge with f_n' converging, i.e., $f_n(x) = n \to \infty$ while $f_n' = 0 \to 0$.

Theorem 6.3.2. Let (f_n) be a sequence of differentiable functions defined on a closed interval $[a, b]$, and assume (f_n') converges uniformly to a function g on $[a, b]$. If there exist a point $x_0 \in [a, b]$ such that $f_n(x_0)$ is a convergent sequence, then (f_n) converges uniformly.

Combining the previous two theorems gives us a better convergence result.

Theorem 6.3.3. Let (f_n) be a sequence of differentiable functions on the closed interval $[a, b]$, and assume (f_n') converges uniformly to a function g on $[a, b]$. If there is a point $x_0 \in [a, b]$ such that $f_n(x_0)$ is a convergent sequence, then (f_n) converges uniformly, and the limit function $f = \lim f_n$ is differentiable with $f' = g$.

Proof. All we need to do is recognize that by Theorem 6.3.2, we have uniform convergence of (f_n) to a function f, which implies that f_n converges pointwise to f.

Then we apply Theorem 6.3.1. $\square$

Example. Consider the sequence of functions

$$f_n(x) = \frac{\ln(1 + nx^2)}{2n}, \quad x \in [1, 2].$$

Here $f_n(1)$ converges to 0, and the sequence of derivatives

$$f_n'(x) = \frac{2nx}{2n(1 + nx^2)} = \frac{x}{1 + nx^2}, \quad x \in [1, 2]$$

converges pointwise to $g(x) = 0$, $x \in [0, 2]$.

The convergence of f_n' to g is uniform on $[1, 2]$ because $|x| \leq 2$ and $1 + nx^2 \geq n$ implies

$$\left| \frac{x}{1 + nx^2} - 0 \right| = \frac{|x|}{1 + nx^2} \leq \frac{2}{n}.$$

Then by Theorem 6.3.3, the sequence (f_n) converges uniformly (something not easily proved directly) and the limit function $f = \lim f_n$ is differentiable with $f' = g$.

Here $f(x) = 0$ for $x \in [1, 2]$ whose derivative is 0.