Math 341 Lecture #33
§7.2: The Definition of the Riemann Integral

We begin by recalling the definition of the Riemann integral.

Starting with a function f on $[a,b]$, we partition $[a,b]$ into subintervals $[x_{k-1},x_k]$, pick sample points $c_k \in [x_{k-1},x_k]$, and form the sum

$$\sum_{k=1}^{n} f(c_k)\Delta x_k$$

where $\Delta x_k = x_k - x_{k-1}$.

We defined the integral of f on $[a,b]$ to be the limit of the Riemann sums as $\max \Delta_k \to 0$, provided the limit exists.

We will approach the Riemann integral differently, using supremums and infimums.

Let f be a bounded function defined on the compact interval $[a,b]$; there is $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a,b]$.

Definition 7.2.1. A *partition* of $[a,b]$ is a finite, ordered set of points

$$P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}.$$

Set $\Delta x_k = x_k - x_{k-1}$ for each $k = 1, 2, \ldots, n$.

For each subinterval $[x_{k-1},x_k]$ of P we set

$$m_k = \inf\{f(x) : x \in [x_{k-1},x_k]\} \text{ and } M_k = \sup\{f(x) : x \in [x_{k-1},x_k]\}.$$

The *lower sum* of f with respect to P is given by

$$L(f,P) = \sum_{k=1}^{n} m_k \Delta x_k.$$

The *upper sum* of f with respect to P is given by

$$U(f,P) = \sum_{k=1}^{n} M_m \Delta x_k.$$

For a given partition, because $m_k \leq M_k$ on each $[x_{k-1},x_k]$ we have that

$$L(f,P) \leq U(f,P).$$

But what happens when we change partitions?

Definition 7.2.2. A partition Q is a *refinement* of a partition P, written $P \subseteq Q$, if Q contains all of the points of P.

Lemma 7.2.3. If $P \subseteq Q$, then $L(f,P) \leq L(f,Q)$ and $U(f,P) \geq U(f,Q)$.

Proof. Let $[x_{k-1},x_k]$ be an subinterval of P.

Suppose there is a point $z \in Q$ such that $x_{k-1} < z < x_k$, so that z "splits" $[x_{k-1}, x_k]$ into two subintervals $[x_{k-1}, z]$ and $[z, x_k]$.

We have the infimum m_k of f on $[x_{k-1}, x_k]$.

Let $m_k' = \inf\{f(x) : x \in [z, x_k]\}$ and $m_k'' = \inf\{f(x) : x \in [x_{k-1}, z]\}$.

Then we have that $m_k \leq m_k'$ and $m_k \leq m_k''$ (the infimum of f over a larger set cannot get bigger), so that

$$\begin{aligned} m_k \Delta x_k &= m_k(x_k - x_{k-1}) \\ &= m_k(x_k - z + z - x_{k-1}) \\ &= m_k(x_k - z) + m_k(z - x_{k-1}) \\ &\leq m_k'(x_k - z) + m_k''(z - x_k). \end{aligned}$$

This implies that the lower sum cannot get smaller when adding more points to a partition.

A similar argument show that the upper sum cannot get bigger when adding more points to a partition. □

We would think it strange if a lower sum for one partition were bigger than an upper sum for another partition. Luckily this cannot happen.

Lemma 7.2.4. If P_1 and P_2 are partitions of $[a, b]$, then $L(f, P_1) \leq U(f, P_2)$.

Proof. We form a third partition and use Lemma 7.2.3.

The partition $Q = P_1 \cup P_2$ is a refinement of P_1 and a refinement of P_2; it is a common refinement of both P_1 and P_2.

Because $P_1 \subseteq Q$ and $P_2 \subseteq Q$, we have that

$$L(f, P_1) \leq L(f, Q) \leq U(f, Q) \leq U(f, P_2)$$

by Lemma 7.2.3. □

This raises the questions: what is the supremum of the lower sums, and what is the infimum of the upper sums, and are they the same?

Definition 7.2.5. Let $\mathcal{P}$ be the collection of all possible partitions of $[a, b]$. The *upper integral* of f on $[a, b]$ is

$$U(f) = \inf\{U(f, P) : P \in \mathcal{P}\}$$

and the *lower sum* of f is

$$L(f) = \sup\{L(f, P) : P \in \mathcal{P}\}.$$

Lemma 7.2.6. For any bounded function f on $[a, b]$, there holds $L(f) \leq U(f)$.

Proof. For partitions P and Q of $[a, b]$ we have by Lemma 7.2.4 that $L(f, P) \leq U(f, Q)$.

So $U(f, Q)$ is an upper bound on the set $\{L(f, P) : P \in \mathcal{P}\}$.

It follows that

$$L(f) = \sup\{L(f, P) : P \in \mathcal{P}\} \leq U(f, Q).$$

This says that $L(f)$ is a lower bound for $U(f, Q)$, so that

$$L(f) \leq \inf\{U(f, Q) : Q \in \mathcal{P}\} = U(f),$$

thus giving the inequality. $\square$

A special situation happens when $L(f) = U(f)$.

Definition 7.2.7 (Riemann Integrability) A bounded function f on the compact interval $[a, b]$ if *Riemann-integrable* if $U(f) = L(f)$. In this case we write

$$\int_a^b f = U(f) = L(f).$$

There are other types of integration besides the Riemann integral, but we will stick to the Riemann integral here, and drop the "Riemann" from Riemann-integrable.

What bounded functions on $[a, b]$ are integrable?

We have the following integrability criterion, which in part will identity which bounded functions are integrable. [The complete answer is in Section 7.6 at the top of page 242.]

Theorem 7.2.8. A bounded function f is integrable on $[a, b]$ if and only if for every $\epsilon > 0$ there exists a partition P_ϵ of $[a, b]$ such that

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.$$

Proof. Suppose for each $\epsilon > 0$ there exists a partition P_ϵ of $[a, b]$ such that $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$.

Since $U(f) \leq U(f, P_\epsilon)$ and $L(f) \geq L(f, P_\epsilon)$ we have that

$$U(f) - L(f) \leq U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.$$

The arbitrariness of $\epsilon > 0$ then implies that $U(f) = L(f)$, and so f is integrable.

Now suppose that f is integrable so that $U(f) = L(f)$.

Let $\epsilon > 0$.

Since $U(f)$ is the infimum of $U(f, P)$ over all partitions P of $[a, b]$, there exists a partition P_1 such that

$$U(f, P_1) < U(f) + \frac{\epsilon}{2}.$$

Since $L(f)$ is the supremum of $L(f, P)$ over all partitions P of $[a, b]$, there exists a partition P_2 such that

$$L(f, P_2) > L(f) - \frac{\epsilon}{2}.$$

For the common refinement $P_\epsilon = P_1 \cup P_2$ and with $U(f) = L(F)$ we have

$$\begin{aligned}
U(f, P_\epsilon) - L(f, P_\epsilon) &\leq U(f, P_1) - L(f, P_2) \\
&= U(f, P_1) - U(f) + L(f) - L(f, P_2) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

which gives the integrability criterion. □

A continuous function f on the compact $[a, b]$ is bounded by the Extreme Value Theorem.

We are now in position to prove that every continuous function on $[a, b]$ is integrable.

Theorem 7.2.9. If f is continuous on the compact $[a, b]$, then f is integrable.

Proof. The continuity of f on the compact $[a, b]$ implies the uniform continuity of f on $[a, b]$.

Thus for $\epsilon > 0$ there exists $\delta > 0$ such that when $|x - y| < \delta$ with $x, y \in [a, b]$, we have

$$|f(x) - f(y)| < \frac{\epsilon}{b-a}.$$

We are driving for the criterion for integrability.

To this end we choose a partition P of $[a, b]$ where $\Delta x_k < \delta$ for all $k = 1, \ldots, n$.

The function f is continuous on the compact subinterval $[x_{k-1}, x_l]$ of P, so we have by the Extreme Value Theorem points $y_k, z_k \in [x_{k-1}, x_k]$ where $f(y_k) = m_k$ and $f(z_k) = M_k$.

Since $y_k, z_k \in [x_{k-1}, x_k]$ we have that

$$M_k - m_k = f(z_k) - f(y_k) < \frac{\epsilon}{b-a}.$$

This implies that

$$U(f, P) - L(f, P) = \sum_{k=1}^{n} (M_k - m_k)\Delta x_k < \frac{\epsilon}{b-a} \sum_{k=1}^{n} \Delta x_k = \epsilon.$$

Therefore, f is integrable by Theorem 7.2.8. □

This raises the question about how discontinuous a bounded function can be and yet still be integrable.