Math 341 Lecture #36
§7.4: Properties of the Integral, Part II

Property (v) of Theorem 7.4.2., $|\int f| \leq \int |f|$, is very useful in analysis, and we will use it to prove that if $f_n \to f$ uniformly with each f_n integrable on $[a,b]$, then f is integrable and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

It is this result that we need to justify integrating a convergence power series term by term to get the integral of the power series.

When we defined the integral of f on $[a,b]$ we tacitly assumed that $a < b$.

Definition 7.4.3. If f is integrable on $[a,b]$, we define

$$\int_b^a = -\int_a^b f.$$

Also we define

$$\int_c^c f = 0.$$

These conventions help with the algebra of integrals.

We can also extend the property

$$\int_a^b f = \int_a^c f + \int_c^b f$$

from $a < c < b$ to any a, b, c in the interval I on which f is integrable.

Now with the basic algebraic properties of the Riemann integral in place, we turn our attention to how the Riemann integral handles sequences of functions.

If (f_n) is a sequence of integrable functions on $[a,b]$, and $f_n \to f$ pointwise, does

$$\int_a^b f_n \to \int_a^b f\ ?$$

It is possible that f need not be integrable even when each f_n is integrable, and so there is no $\int_a^b f$ to which $\int_a^b f_n$ can converge (see homework problem 7.3.7).

What else can go wrong?

Example. For each $n \in \mathbb{N}$, the function $f_n : [0,1] \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} n & \text{if } 0 < x < 1/n, \\ 0 & \text{if } x = 0 \text{ or } 1/n \leq x \leq 1, \end{cases}$$

has two discontinuities and so is integrable with

$$\int_0^1 f_n = 1.$$

The pointwise limit of f_n is $f(x) = 0$, $x \in [0, 1]$, for which

$$\int_0^1 f = 0.$$

Thus we have that

$$\int_0^1 f_n \to 1 \neq \int_0^1 f = 0.$$

Pointwise convergence is insufficient to get the result.

Theorem 7.4.4. Let f_n be a sequence of functions defined on $[a, b]$ converging pointwise to f on $[a, b]$. If $f_n \to f$ uniformly on $[a, b]$, and each f_n is integrable on $[a, b]$, then f is integrable and

$$\int_a^b f_n \to \int_a^b f.$$

Proof. We showed in a Homework Problem (Exercise 7.2.5) that if each f_n is integrable and $f_n \to f$ uniformly that f is integrable.

To show the convergence of the integrals, we consider

$$\left| \int_a^b f_n - \int_a^b f \right|$$

which we want to show is smaller than ϵ for all sufficiently large n.

Since f_n and f are integrable, then $f_n - f$ is integrable, and by the algebraic rules of integration we have

$$\begin{aligned} \left| \int_a^b f_n - \int_a^b f \right| &= \left| \int_a^b (f_n - f) \right| \\ &\leq \int_a^b |f_n - f|. \end{aligned}$$

By the uniform convergence of f_n to f, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$|f_n(x) - f(x)| < \frac{\epsilon}{b - a}$$

for all $x \in [a, b]$.

Thus we have for all $n \geq N$ that

$$\int_a^b |f_n - f| \leq \int_a^b \frac{\epsilon}{b - a} = \epsilon.$$

Putting it all together we have for every $\epsilon > 0$ the existence of $N \in \mathbb{N}$ such that for all $n \geq N$ there holds

$$\left| \int_a^b f_n - \int_a^b f \right| < \epsilon.$$

This says that the sequence of numbers $\int_a^b f_n$ converges to the number $\int_a^b f$, or that

$$\int_a^b f_n \to \int_a^b f$$

in other words. $\square$

Recall that we integrated a power series term by term to get the integral of the power series.

We can now prove that this is true.

Theorem. Let A be the interval of convergence for a power series $\sum_{k=0}^{\infty} a_k x^k$. Then for $[a, b] \subseteq A$, we have

$$\int_a^b \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \int_a^b a_k x^k.$$

Proof. The interval $[a, b] \subseteq A$ is compact, so that by Theorem 6.5.5, the power series converges uniformly on $[a, b]$.

Each partial sum $f_n(x) = \sum_{k=0}^{n} a_k x^k$ is integrable because it is a polynomial.

By Theorem 7.4.4 the power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ is integrable on $[a, b]$ and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

Set

$$b_k = \int_a^b a_k x^k, \ s_n = \sum_{k=0}^{n} b_k.$$

Then

$$\begin{aligned}
\int_a^b \sum_{k=0}^{\infty} a_k x^k &= \int_a^b f \ \ [\text{Definition of } f] \\
&= \lim_{n\to\infty} \int_a^b f_n(x) \ \ [\text{Theorem 7.4.4}] \\
&= \lim_{n\to\infty} \int_a^b \sum_{k=0}^{n} a_k x^k \ \ [\text{Definition of } f_n] \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \int_a^b a_k x^k \ \ [\text{Integral of Finite sum}] \\
&= \lim_{n\to\infty} s_n \ \ [\text{Defintion of } s_n].
\end{aligned}$$

Hence the sequence of partial sums (s_n) converges, giving

$$\int_a^b \sum_{k=0}^{\infty} a_k x^k = \lim_{n\to\infty} s_n = \sum_{k=0}^{\infty} b_k = \sum_{k=0}^{\infty} \int_a^b a_k x^k,$$

which is the desired result. $\square$