## Math 341 Lecture \#37

## §7.5: The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus asserts that differentiation and integration have an remarkable inverse relationship.

You have seen the Fundamental Theorem of Calculus and its proof in the context of continuous functions.
Fundamental Theorem of Calculus in Math 112. (i) If $f:[a, b] \rightarrow \mathbb{R}$ is continuous, and $F:[a, b] \rightarrow \mathbb{R}$ is a differentiable function that satisfies $F^{\prime}(x)=f(x)$ for all $x \in[a, b]$, then

$$
\int_{a}^{b} f(x) d x=F(b)-F(a) .
$$

(ii) Let $g:[a, b] \rightarrow \mathbb{R}$ be continuous, and define

$$
G(x)=\int_{a}^{x} g(t) d t
$$

for all $x \in[a, b]$. Then $G$ is continuous on $[a, b]$, differentiable on $[a, b]$, and satisfies $G^{\prime}(x)=g(x)$ for all $x \in[a, b]$.

We will prove this in the context of integrable functions, which will require much of the theory we have developed to date.
Theorem 7.5.1 (Fundamental Theorem of Calculus). (i) If $f:[a, b] \rightarrow \mathbb{R}$ is integrable, and $F:[a, b] \rightarrow \mathbb{R}$ is a differentiable function that satisfies $F^{\prime}(x)=f(x)$ for all $x \in[a, b]$, then

$$
\int_{a}^{b} f=F(b)-F(a)
$$

(ii) Let $g:[a, b] \rightarrow \mathbb{R}$ be integrable, and define

$$
G(x)=\int_{a}^{x} g
$$

for all $x \in[a, b]$. Then $G$ is continuous on $[a, b]$. If $g(x)$ is continuous at some point $c \in[a, b]$, then $G$ is differentiable at $c$ and $G^{\prime}(c)=g(c)$.
Proof. (i) Let $P$ be a partition of $[a, b]$.
For a subinterval $\left[x_{k-1}, x_{k}\right]$ of $P$ we have by Mean Value Theorem the existence of $t_{k} \in$ $\left(x_{k-1}, x_{k}\right)$ for which

$$
F\left(x_{k}\right)-F\left(x_{k-1}\right)=F^{\prime}\left(t_{k}\right)\left(x_{k}-x_{k-1}\right)=f\left(t_{k}\right)\left(x_{k}-x_{k-1}\right) .
$$

On the subinterval $\left[x_{k-1}, x_{k}\right]$ we have $m_{k} \leq f\left(t_{k}\right) \leq M_{k}$, so that

$$
L(f, P)=\sum_{k=1}^{n} m_{k} \Delta x_{k} \leq \sum_{k=1}^{n}\left[F\left(x_{k}\right)-F\left(x_{k-1}\right)\right] \leq \sum_{k=1}^{n} M_{k} \Delta x_{k}=U(f, P)
$$

The sum in the middle telescopes:

$$
\sum_{k=1}^{n}\left[F\left(x_{k}\right)-F\left(x_{k-1}\right)\right]=F(b)-F(a)
$$

which is independent of the partition $P$.
Then we have for all partitions $P$ of $[a, b]$ that

$$
L(f, P) \leq F(b)-F(a) \leq U(f, P)
$$

Thus we have that $L(f) \leq F(b)-F(a) \leq U(f)$.
Since $f$ is integrable on $[a, b]$ we have that $L(f)=\int_{a}^{b} f=U(f)$, so that

$$
\int_{a}^{b} f=F(b)-F(a)
$$

(ii) Let $M$ be a bound on the integrable function $g$ : we have $|g(x)| \leq M$ for all $x \in[a, b]$. For $x, y \in[a, b]$, with $y<x$, we have that

$$
|G(x)-G(y)|=\left|\int_{a}^{x} g-\int_{a}^{y} g\right|=\left|\int_{y}^{x} g\right| \leq \int_{y}^{x}|g| \leq M|x-y| .
$$

A similar inequality holds if $x<y$, and so we get that

$$
\left|\frac{G(x)-G(y)}{x-y}\right| \leq M
$$

for all distinct $x, y \in[a, b]$, which says that $G$ is Lipschitz.
In a Homework Problem (Exercise 4.4.9) you showed that Lipschitz implies uniform continuity, and so $G$ is uniformly continuous on $[a, b]$.
Hence $G$ is continuous on $[a, b]$.
Now suppose that $g$ is continuous at some $c \in[a, b]$.
We are to show that $G$ is differentiable at $c$ and $G^{\prime}(c)=g(c)$ : we will show both at the same time.
We thus want to show that

$$
\begin{aligned}
\lim _{x \rightarrow c} \frac{G(x)-G(c)}{x-c} & =\lim _{x \rightarrow c} \frac{1}{x-c}\left[\int_{a}^{x} g-\int_{a}^{c} g\right] \\
& =\lim _{x \rightarrow c} \frac{1}{x-c}\left[-\int_{x}^{a} g-\int_{a}^{c} g\right]=\frac{1}{x-c}\left[-\int_{x}^{c} g\right] \\
& =\lim _{x \rightarrow c} \frac{1}{x-c} \int_{c}^{x} g .
\end{aligned}
$$

exists and equals $g(c)$.

In other words, for $\epsilon>0$ we want to find $\delta>0$ such that when $0<|x-c|<\delta$ we have

$$
\left|\frac{1}{x-c}\left(\int_{c}^{x} g\right)-g(c)\right|<\epsilon .
$$

We rewrite the constant $\alpha=g(c)$ as an integral,

$$
g(c)=\frac{1}{x-c} \int_{c}^{x} \alpha
$$

so that

$$
\begin{aligned}
\left|\frac{1}{x-c}\left(\int_{c}^{x} g\right)-g(c)\right| & =\left|\frac{1}{x-c}\left(\int_{c}^{x} g\right)-\frac{1}{x-c}\left(\int_{c}^{x} \alpha\right)\right| \\
& =\left|\frac{1}{x-c} \int_{c}^{x}(g-\alpha)\right| \\
& =\left|\frac{1}{x-c} \int_{c}^{x}(g(t)-g(c)) d t\right| \\
& \leq \frac{1}{x-c} \int_{c}^{x}|g(t)-g(c)| d t
\end{aligned}
$$

You might be wondering why there is not absolute value on the $x-c$ term.
Well, when $x-c>0$ there is no problem, but if $x-c<0$, then the integral from $c$ to $x$ is nonpositive, and thus the product is nonnegative as well.

How do we control the integral of $|g(t)-g(c)|$ ?
We have assumed that $g$ is continuous at $c$, so for $\epsilon>0$ there is $\delta>0$ such that $|g(y)-g(c)|<\epsilon$ whenever $|y-c|<\delta$.
Restricting $x$ to the interval $|x-c|<\delta$ and keeping in mind that $|x-c| \geq|t-c|$ for $t$ in the integrand, we have

$$
\frac{1}{x-c} \int_{c}^{x}|g(t)-g(c)| d t \leq \frac{1}{x-c} \int_{c}^{x} \epsilon d t=\epsilon
$$

Thus we have shown that $G$ is differentiable at $c$ and $G^{\prime}(c)=g(c)$.

