Final Exam Review Solutions, Math 341

Winter 2014

DISCLAIMER: The solutions provided are not necessarily the unique solution to the problem.

1. Arrange the sets of points in an array as follows:

$$\begin{array}{l}(1,1)(1,2)(1,3)(1,4)\cdots\\(2,1)(2,2)(2,3)(2,4)\cdots\\(3,1)(3,2)(3,3)(3,4)\cdots\\(4,1)(4,2)(4,3)(4,4)\cdots\end{array}$$

   Then establish a bijection between $\mathbb{N}$ and the elements of the set as follows:

$$\begin{array}{l}1\mapsto(1,1)\\2\mapsto(2,1)\\3\mapsto(1,2)\\4\mapsto(3,1)\\5\mapsto(2,2)\end{array}$$

   and continue in a similar fashion, moving along parallel diagonal lines. This gives the required set is countable.

2. The limit is certainly zero. To prove this, given $\varepsilon > 0$, set $N > \sqrt{\frac{2}{\varepsilon}}$. Then, for $n > N$, we have

$$|a_n - 0| = \left|\frac{\sin(n)+1}{n^2}\right| \leq \left|\frac{\sin(n)}{n^2}\right| + \left|\frac{1}{n^2}\right| \leq \left|\frac{1}{n^2}\right| + \left|\frac{1}{n^2}\right| = \left|\frac{2}{n^2}\right| < \frac{2}{\left(\sqrt{\frac{2}{\varepsilon}}\right)^2} = \varepsilon$$

   as needed.

3. This converges by comparison:

$$\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^3-2} \leq \sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^3} = \sum_{n=1}^{\infty}\frac{1}{n^{2.5}}$$

   which converges by the p-test.

4. Let $c \in A$ be arbitrary. Then $f(c) > 0$. Set $\varepsilon = f(c)$. Then, since $f(x)$ is continuous, there is some δ so that if $|x-c| < \delta$, then $|f(x) - f(c)| < \varepsilon/2$. So $|x-c| < \delta$ implies that $f(x)$ is within $\varepsilon/2$ of $f(c)$, so $f(x)$ must also be positive. So the entire interval $|x-c| < \delta$ must be contained in A as well. So every point $c \in A$ has an interval about it contained in A. Hence, A is open.

5. Since P and Q are perfect, they are both closed and contain no isolated points. The finite union of closed sets is closed, so $P \cup Q$ is also closed. Also, for any point x of $P \cup Q$, either $x \in P$ or $x \in Q$. Without loss of generality, suppose $x \in P$. Since P contains no isolated points, $V_\varepsilon(x)$ must contain a point of P other than x. So $V_\varepsilon(x)$ must contain a point of $P \cup Q$ other than x, so $P \cup Q$ contains no isolated points. Hence, $P \cup Q$ is perfect.

For the second part, note that one perfect set by itself is trivially perfect, and for perfect sets P_1 and P_2, we know that $P_1 \cup P_2$ is perfect by the above argument. Suppose now inductively that

$$P_1 \cup \ldots \cup P_n$$

is perfect. We wish to show the union of $n+1$ perfect sets is perfect. We then have that

$$P_1 \cup \ldots \cup P_n \cup P_{n+1} = (P_1 \cup \ldots \cup P_n) \cup P_{n+1}$$

which is the union of two perfect sets, and is therefore perfect by the first argument.

Lastly, consider the sets $P_n = [0, 2 - \frac{1}{n}]$. Each of these is certainly perfect, but their union is $[0,2)$, which is not.

6. We define two sequences:

$$x_n = \left\{ \frac{1}{\pi/2}, \frac{1}{5\pi/2}, \frac{1}{9\pi/2}, \ldots \right\},$$

$$y_n = \left\{ \frac{1}{3\pi/2}, \frac{1}{7\pi/2}, \frac{1}{11\pi/2}, \ldots \right\}.$$

Then $\sin(1/x_n) = 1$ for all n, and $\sin(1/y_n) = -1$ for all n. Since both x_n and y_n converge to zero, and

$$\lim_{n\to\infty} \sin\left(\frac{1}{x_n}\right) \neq \lim_{n\to\infty} \sin\left(\frac{1}{y_n}\right),$$

we know that the limit as $x \to 0$ does not exist.

7. By the mean value theorem, for two points x and y where (WLOG) $x < y$, we have

$$f'(c) = \frac{f(y) - f(x)}{y - x}$$

for some $c \in (x, y)$. We can compute

$$f'(x) = 1 - \sin(x)$$

, and so we know that $|f'(x)| \leq 2$ for all x. We must then have

$$\left| \frac{f(y) - f(x)}{y - x} \right| \leq 2,$$

which gives

$$|\frac{f(y) - f(x)}{|} \leq 2|y - x|.$$

Now, given $\varepsilon > 0$, set $\delta = \varepsilon/2$. Then if $|y - x| < \delta$,

$$|f(y) - f(x)| \leq 2|y - x| < 2\delta = 2\frac{\varepsilon}{2} = \varepsilon.$$

8. First, we show that a root exists. Since $f(x)$ is a polynomial, it is continuous, so the intermediate value theorem applies. We find that $f(-1) = -1$ and $f(0) = 1$, so there is some $c \in (-1, 0)$ which satisfies $f(c) = 0$. So there is at least one root of the equation.

Next, suppose there are two (or more) roots of the equation. Then there exist x_1 and x_2 with $f(x_1) = f(x_2) = 0$ and $x_1 \neq x_2$. Then, since $f(x)$ is differentiable, the mean value theorem applies. So we must have that

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} = f'(c)$$

for some c between x_1 and x_2. Since $f(x_1) = f(x_2) = 0$, this means that $f' = 0$ at some point. But we know that $f'(x) = 5x^4 + 1$ can never be zero, so two or more roots are impossible. So there can only be one root.

9. Since differentiability implies continuity, the only point at which $f(x)$ can be differentiable is at $x = 0$, because $f(x)$ is discontinuous at every other point. To compute $f'(0)$, we use the definition

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim x \to 0 \frac{f(x)}{x}.$$

From the definition of f, we know that

$$\lim x \to 0 \frac{0}{x} \leq \lim x \to 0 \frac{f(x)}{x} \leq \lim x \to 0 \frac{x^2}{x}.$$

Both of the outer limits are zero, so by the squeeze theorem (or order limit theorem applied twice), we know that the center limit must be zero as well, so $f'(0) = 0$.

10. Notice that for any fixed value of x, we know that $x/n \to 0$ as $n \to \infty$. So x/n lies within the interval $(-1, 1)$ for sufficiently large n. But if $x/n \in (-1, 1)$, then $|x/n|^n \leq |x/n|$ for any $n \in \mathbb{N}$. This means that

$$\lim_{n \to \infty} 0 \leq \lim_{n \to \infty} |x/n|^n \leq \lim_{n \to \infty} |x/n|.$$

Since the outer two limits go to zero, it must be true that $|x/n|^n$ approaches zero as well. But this means that $(x/n)^n$ must approach zero as $n \to \infty$. So $f_n(x)$ approaches the function $f(x)$ that is zero everywhere.

This convergence is not uniform. In order for the convergence to be uniform, we would need to produce a natural number N so that for a given $\varepsilon > 0$, $|f_n(x) - 0| < \varepsilon$ for $n > N$. But each function $f_n(x)$ has arbitrarily large points for large enough x. So it is not possible to get a uniform bound on the functions, so the convergence is not uniform.

11. First, we note that

$$f_n'(x) = \frac{1 - nx^2}{(1 + nx^2)^2},$$

and that $f_n'(x) = 0$ when $x = \frac{1}{\sqrt{n}}$. Moreover, $f_n'(x) > 0$ when $x < \frac{1}{\sqrt{n}}$ and $f_n'(x) < 0$ when $x > \frac{1}{\sqrt{n}}$, so the value of $f_n(x)$ at $x = \frac{1}{\sqrt{n}}$ is a maximum. We find that $f_n(1/\sqrt{n}) = \frac{1}{2\sqrt{n}}$ for any given n.

By inspection, we guess that $f_n(x)$ will converge to the zero function. To show this, given $\varepsilon > 0$, set $N = 1/4\varepsilon^2$. Then for $n > N$, we have

$$|f_n(x) - 0| = \left| \frac{x}{1 + nx^2} \right| \leq \left| \frac{1}{2\sqrt{n}} \right| < \left| \frac{1}{\frac{2}{2\varepsilon}} \right| = \varepsilon$$

as needed.

Next, from the definition of $f_n'(x)$ we have above, expanding the denominator gives us

$$f_n'(x) = \frac{1-nx^2}{1+2nx^2+n^2x^4}.$$

Since n has a higher power in the denominator, we know that $f_n'(x)$ will approach zero for any x as $n \to \infty$. So the limit function $g(x) = 0$ for all x.

Lastly, since $f(x) = 0$, we have that $f'(x) = 0$, and so $f' = g$ in this case.

NOTE: This generally does not have to be true. Review homework problem 6.3.2 and Theorem 6.3.3 for a bit more detail on this.

12. We use the ratio test to find the interval of convergence. The calculation is as follows:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0.$$

Since this limit is less than 1 for all x, we know that the series converges everywhere.

Next, let K be a compact set in $\mathbb{R}$. Then K is closed and bounded, so there is some R so that $|x| < R$ for all $x \in K$. Set $M_n = \frac{R^n}{n!}$. Then, for any $x \in K$, we know that

$$\frac{x^n}{n!} \leq M_n.$$

Since

$$\sum_{n=0}^{\infty} M_n$$

converges (use the above result or use the ratio test again), then by the Weirstrass M-test, the power series converges uniformly on K.

13. We start with a known power series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Substituting $x = -y^2$ gives

$$\frac{1}{1+y^2} = \sum_{n=0}^{\infty}(-y^2)^n = \sum_{n=0}^{\infty}(-1)^n y^{2n}.$$

We recognize the left term as the derivative of $\tan^{-1}(y)$. We integrate with respect to y to find

$$\tan^{-1}(y) = \sum_{n=0}^{\infty}\frac{(-1)^n y^{2n+1}}{2n+1} + C$$

for some constant C. We can find C immediately by plugging in $y = 0$. This gives $0 = 0 + C$, and so $C = 0$. Changing the variable back to x gives the needed power series:

$$\tan^{-1}(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}.$$

To find the interval of convergence, we again apply the ratio test:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+3}}{2n+3}\cdot\frac{2n+1}{(-1)^n x^{2n+1}}\right| = |x^2|.$$

If we require this to be less than 1, we get the interval $(-1,1)$. The final remaining step is to check the endpoints. For this, if we plug in $x=1$, we get the sum

$$\sum_{n=0}^{\infty}\frac{(-1)^n 1^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

which converges by the alternating series test. For $x=-1$, since $2n+1$ is always odd, we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n(-1)^{2n+1}}{2n+1} = -\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1},$$

which converges again by the alternating series test. So the interval of convergence is $[-1,1]$.

14. *A picture really helps for this – you should draw the sine curve here.*

To compute the upper sums, note that on the intervals $[0,\pi/3]$ and $[2\pi/3,\pi]$, the maximum value of $f(x)$ is $\sqrt{3}/2$, and on $[\pi/3,2\pi/3]$, the maximum value is 1. So $U(f,P)=(\pi/3)(\sqrt{3}/2+1+\sqrt{3}/2)$. Also (see picture), we find the minimum values of f on the intervals $[0,\pi/3]$ and $[2\pi/3,\pi]$ are zero, and the minimum value of f on $[\pi/3,2\pi/3]$ is $\sqrt{3}/2$, so $L(f,P)=(\pi/3)(0+\sqrt{3}/2+0)$.

Notice that P' is a refinement of P, and so we will have that

$$U(f,P') \le U(f,P) \text{ and } L(f,P') \ge L(f,P).$$

15. To compute this, let $f(x)=e^{-x^2}$, and let $F(x)$ be the antiderivative (so $F'=f$). Even though we can't directly compute F, we can evaluate the integral in terms of F. This gives us

$$\frac{d}{dx}\int_{x^2}^{x^3} e^{-t^2}\,dt = \frac{d}{dx}\left(F(x^3)-F(x^2)\right).$$

Now, using the chain rule, we find the answer:

$$\frac{d}{dx}\left(F(x^3)-F(x^2)\right) = f(x^3)\cdot 3x^2 - f(x^2)\cdot 2x = 3x^2e^{-x^6} - 2xe^{-x^4}.$$

16. The statement is false. A good counterexample comes by letting $I=[0,1]$ and $f_n(x)=x^n$. Then we can compute

$$\int_0^1 f_n(x)\,dx = \int_0^1 x^n\,dx = x^{n+1}|_0^1 = \frac{1}{n+1},$$

and so

$$\int_0^1 f_n(x)\,dx \to 0.$$

However, we know that $f_n(x)$ converges to a function f that is zero on $[0,1)$ and one when $x=1$, so $f(x)\neq 0$ for all x.

17. The statement is false. An easy counterexample is to let the function $f(x)=-e^x$. Then $f(x)<0$ for all x. By the fundamental theorem of calculus, we have that $g'(x)=f(x)<0$ for all x, so g is actually decreasing.