Final Exam Review, Math 341

Winter 2014

1. Show that the set of all points
$$\{(x,y)\ :\ x,y\in\mathbb{N}\}$$
is countable.

2. Define
$$a_n = \frac{\sin(n)+1}{n^2}.$$
Find
$$\lim_{n\to\infty} a_n,$$
and prove this limit is correct using the definition.

3. Determine whether
$$\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^3-2}$$
converges or diverges.

4. Suppose a function $f(x):\mathbb{R}\to\mathbb{R}$ is continuous. Define the set
$$A=\{x\in\mathbb{R}: f(x)>0\}.$$
Show that A is open.

5.
   - Let P and Q be perfect sets. Prove that $P\cup Q$ is perfect.
   - Show that the finite union of perfect sets is perfect.
   - Give an example to show that the infinite union of perfect sets may not be perfect.

6. Show that
$$\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$$
does not exist.

7. Show that
$$f(x)=x+\cos(x)$$
is uniformly continuous on all of $\mathbb{R}$.

8. Show that the function
$$f(x)=x^5+x+1$$
has exactly one root. That is, show that there is only one point x_0 such that $f(x_0)=0$.

9. For the function defined by
$$f(x)=\begin{cases} x^2 & : x\in\mathbb{Q}\\ 0 & : x\notin\mathbb{Q}\end{cases}$$
show that $f(x)$ is differentiable at exactly one point, and evaluate the derivative at that point.

10. Let

$$f_n(x) = \left(\frac{x}{n}\right)^n.$$

Show that $f_n(x)$ converges pointwise to the function $f(x) = 0$ on all of $\mathbb{R}$. Is this convergence uniform?

11. Let

$$f_n(x) = \frac{x}{1+nx^2}$$

be defined on $x \in [0,1]$.

- Show $f_n(x)$ converges uniformly to a function $f(x)$.
- Show that $f_n'(x)$ converges to a function $g(x)$.
- Show that $f'(x) = g(x)$.

12. Show that

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges on all of $\mathbb{R}$, and that it converges uniformly on any compact subset of $\mathbb{R}$.

13. Find the power series for $\arctan x$, and find its interval of convergence.

14. For the function

$$f(x) = \sin(x)$$

and

$$P = \left\{0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi\right\},$$

- Compute $U(f,P)$ and $L(f,P)$.
- Without calculating, what happens when the points $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ are added to P, forming a new partition P'?

15. Compute

$$\frac{d}{dx}\int_{x^2}^{x^3} e^{-t^2}\,dt.$$

Explain why each step is valid.

16. Prove or give a counterexample: If f_n is a sequence of functions on an interval $I = [a,b]$ such that $f_n \to f$, and

$$\int_a^b f_n \to 0 \text{ as } n \to \infty,$$

then $f(x) = 0$ for all x.

17. Let $f(x)$ be a function that is monotonically increasing on the interval $[a,b]$. Define a function

$$g(x) = \int_a^x f(t)\,dt$$

for $x \in [a,b]$. Is $g(x)$ monotonically increasing? Prove or give a counterexample.