Math 113 Final Exam Practice

The Final Exam is comprehensive. You should refer to prior reviews when studying material in chapters 6, 7, 8, and 11.1-9. This review will cover 11.10-11 and chapter 10. This sheet has three sections. The first section will remind you about techniques and formulas that you should know. The second gives a number of practice questions for you to work on. The third section gives the answers of the questions in section 2.

Review

Finding sums of series - Additional information

Finding a power series that represents a specific function is the next topic. The first one we learned was the geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ iff } x \in (-1,1).$$

We then found the sum of several series by differentiating, integrating, multiplying by x, etc.

The Taylor series of a function is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

and can also be used to find the power series of a function.

Notice that the interval of convergence of these series is still very important. We need to know when we can trust them.

In addition to the geometric series above, the following Maclaurin series (with interval of convergence) are important:

- $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$, $[-1,1]$
- $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$, $(-1,1]$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $(-\infty,\infty)$
- $(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n$, $(-1,1)$ where the binomial coefficients are $\binom{r}{n} = \frac{r(r-1)\cdots(r-n+1)}{n!}$
- $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, $(-\infty,\infty)$
- $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, $(-\infty,\infty)$
- $\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$, $(-\infty,\infty)$
- $\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$, $(-\infty,\infty)$

If you need to construct a Maclaurin series of a function and some of the above functions are included, it is almost always easier to manipulate the Maclaurin series instead of constructing the series by scratch.

Approximating sums of series

In addition to finding whether sums of series converge or not, we also were able to find approximations to the error. There were 3 basic approximations to the error given by the Integral test, Alternating Series test, and the Taylor Series.

1. If $\sum a_k$ is convergent with sum s and $f(k) = a_k$ where f is a continuous, positive, and decreasing function for $x \geq n$, then the remainder $R_n = s - s_n = \sum_{k=n+1}^{\infty} a_k$ satisfies the inequality

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

2. If $\{a_k\}$ is a positive decreasing sequence with a limit of 0, then $\sum (-1)^k a_k$ is convergent with sum s and the remainder $R_n = s - s_n = \sum_{k=n+1}^{\infty} (-1)^k a_k$ satisfies the inequality

$$|R_n| < a_{n+1}$$

3. **Taylor's Inequality**: If $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$ is the nth Taylor polynomial of $f(x)$ centered at c, then the remainder $R_n(x) = f(x) - T_n(x)$ satisfies the inequality
$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-c|^{n+1}$$
on the interval where $|f^{(n+1)}(x)| < M$.

We use this information, when applicable, to find maximum errors when approximating a function by a Taylor polynomial as well.

10.1 Parametric Equations

We learned how to define curves parametrically. That is, we learned how to describe a curve given by an equation
$$H(x,y) = 0$$
in terms of a pair of functions
$$x = f(t), \ y = g(t).$$

You will need to be able to do the following:

(a) Graph a curve from its parametric equations.

(b) Recognize the curve of a set of parametric equations.

(c) Eliminate the parameter of the parametric equations to find an equation in x and y describing the curve.

(d) Construct a set of parametric equations for a curve written in cartesian coordinates.

10.2 Calculus of Parametric Equations

In the discussion below, we will assume that a curve can be described parametrically by
$$x = f(t),$$
$$y = g(t).$$

Slopes
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}.$$
This gives a formula for the slope as a function of parameter.
$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Arclength
$$s = \int_{t_0}^{t_1} \sqrt{(f'(x))^2 + (g'(x))^2}\, dx$$

Surface Area
Rotated about the x axis:
$$S = \int_{t_0}^{t_1} 2\pi f(x)\sqrt{(f'(x))^2 + (g'(x))^2}\, dx$$
Rotated about the y axis:
$$S = \int_{t_0}^{t_1} 2\pi g(x)\sqrt{(f'(x))^2 + (g'(x))^2}\, dx$$

Area under the curve
Area between the curve and the x axis:
$$A = \int_{t_0}^{t_1} y\, dx = \int_{t_0}^{t_1} g(t)f'(t)\, dt$$
Area between the curve and the y axis:
$$A = \int_{t_0}^{t_1} x\, dy = \int_{t_0}^{t_1} f(t)g'(t)\, dt$$

10.3 Polar Coordinates

In this section, we learned how to write points and equations in polar coordinates.

You will need to be able to do the following:

(a) Convert a point from cartesian coordinates to polar coordinates and vice-versa.

(b) Be able to write a polar curve in cartesian coordinates and a cartesian curve in polar coordinates.

(c) Be able to graph and recognize polar curves.

You will need to know the following formulas:

- $x = r\cos\theta$
- $y = r\sin\theta$
- $r = \sqrt{x^2 + y^2}$
- $\theta = \tan^{-1}(y/x)$ if (x, y) is in quadrants 1 or 4. Otherwise, $\theta = \tan^{-1}(y/x) + \pi$.

You will also need to be able to find the slope of a polar curve. Fortunately, we can do this using the standard parameterization of a polar curve. If $r = f(\theta)$ is a polar curve, then from the above equations we can write

$$x = f(\theta)\cos\theta$$

$$y = f(\theta)\sin\theta.$$

Then we can use the techniques of section 10.2 to find the slope of the tangent line.

10.4 Calculus of Polar coordinates

Note that we can use the parameterization of polar curves mentioned in the previous section to find arclength also. However, in this case, the formula simplifies considerably, so it is better to use the simplified formula directly. If $r = f(\theta)$, then the arclength is given by

$$s = \int_{\theta_0}^{\theta_1} \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta.$$

We also wish to find area underneath polar curves. However, since polar curves are defined by angle, underneath really translates to "between the curve and the origin". The area "inside" a polar curve $r = f(\theta)$, or between the polar curve and the origin is given by

$$A = \int_{\theta_0}^{\theta_1} \frac{1}{2} f^2(\theta)\, d\theta.$$

Questions

Try to study the review notes and memorize any relevant equations **before** trying to work these equations. If you cannot solve a problem without the book or notes, you will not be able to solve that problem on the exam.

1. Find the Maclaurin series for $f(x) = \ln(2-x)$ from the definition of a Maclaurin series. Find the radius of convergence.

2. Find a Taylor series for $f(x) = \cos(\pi x)$ centered at $x = 1$. Prove that the series you find represents $\cos(\pi x)$ for all x.

3. Use multiplication to find the first 4 terms of the Maclaurin series for $f(x) = e^x \cosh(2x)$.

4. Use division to find the first 3 terms of the Maclaurin series for $g(x) = \frac{x^2}{\cos x - 1}$.

5. Use the power series of $\frac{1}{\sqrt[3]{1+x}}$ to estimate $\frac{1}{\sqrt[3]{1.1}}$ correct to the nearest 0.0001. Justify that the error is less than 0.0001 using the Alternating Series Estimation Theory or Taylor's Inequality.

6. Find the sum:

   (a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(\sqrt{3})^{2n+1}(2n+1)}$

   (b) $\sum_{n=2}^{\infty} \frac{3}{2^n n!}$

   (c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$

   (d) $\frac{4}{2!} + \frac{8}{3!} + \frac{16}{4!} + \frac{32}{5!} + \ldots$

7. Find the Taylor polynomial $T_3(x)$ for the function $f(x) = \arcsin x$, at $a = 0$.

8. Approximate f by a Taylor polynomial with degree n at the number a. And use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the given interval.
   (a) $f(x) = \sqrt[3]{x}, \quad a = 8, \quad n = 2, \quad 7 \le x \le 9$
   (b) $f(x) = x \sin x, \quad a = 0, \quad n = 4, \quad -1 \le x \le 1$

9. Find the Taylor polynomial $T_3(x)$ for the function $f(x) = \cos x$ at the number $a = \pi/2$. And use it to estimate $\cos 80^0$ correct to five decimal places.

10. A car is moving with speed $20 m/s$ and acceleration $2 m/s^2$ at a given instant. Using a second-degree Taylor polynomial, estimate how far the car moves in the next second. Would it be reasonable to use this polynomial to estimate distance traveled during the next minute?

11. Show that T_n and f have the same derivatives at a up to order n.

    In problems 12 to 14, graph the parametric curve.

12. $x(t) = \cos t$, $y(t) = \sin(2t)$.

13. $x(t) = e^{2t}$, $y(t) = \ln(t) + 1$.

14. $x(t) = \sqrt{t}$, $y(t) = t^{3/2} - 2t$.

    In problems 15 to 17, eliminate the parameter to find a Cartesian equation of the curve.

15. $x(t) = \cos t$, $y(t) = \sin(2t)$.

16. $x(t) = e^{2t}$, $y(t) = \ln(t) + 1$.

17. $x(t) = \sqrt{t}$, $y(t) = t^{3/2} - 2t$.

    In problems 18 to 19, find parametric equations for the curve

18. $x^2 + \frac{y^2}{4} = 1$

19. $y = x^2 + 2x - 1$

20. Find an equation of the tangent to the curve at the given point.

    $$x = \cos(3\theta) + \sin(2\theta), \quad y = \sin(3\theta) + \cos(2\theta); \quad \theta = 0$$

21. For which values of t is the tangent to curve horizontal or vertical? Determine the concavity of the curve.

    $$x = t^2 - t - 1, \quad y = 2t^3 - 6t - 1$$

22. Find the area enclosed by the curve $x = t^2 - 2t, y = \sqrt{t}$ and the y-axis.

23. Find the area of one quarter of the ellipse described by $x = 5\sin(t), y = 2\cos(t)$.

24. Find the exact length of the curve: $x = \frac{t}{1+t}, y = \ln(1+t); \quad 0 \le t \le 2$.

25. Find the exact length of the curve: $x = e^t + e^{-t}, y = 5 - 2t; \quad 0 \le t \le 3$.

26. Find the exact surface area by rotating the curve about the x-axis: $x = t^3, y = t^2; \quad 0 \le t \le 1$.

27. The Cartesian coordinates for a point are $(-1, -\sqrt{3})$. Find polar coordinates (r, θ) for the point where $r > 0$ and $0 \le \theta < 2\pi$.

28. Find the distance between the points with polar coordinates $(2, \pi/3)$ and $(4, 2\pi/3)$.

29. Find a polar equation for the curve represented by the Cartesian equation $x^2 + y^2 = 9$.

30. Identify the curve given in polar coordinates by $r = 4\sin\theta$ by finding a Cartesian equation for the curve.

31. Graph in polar coordinates $r = 2\cos(3\theta)$.

32. Graph in polar coordinates $r = \sin(2\theta)$.

33. Graph in polar coordinates $r = 1 + \cos\theta$.

34. Graph in polar coordinates $r^2 = \cos(2\theta)$.

35. Find the area inside the circle $r = 6\sin\theta$ and outside the limaçon $r = 2 + 2\sin\theta$.

36. Find the area of one petal of the rose given by $r = \cos 3\theta$.

37. Find the length of the polar curve given by $r = \theta$ for $\theta \in [0, \pi]$.

38. Set up but do not evaluate an integral for the length of the polar curve given by $r = \theta + \sin\theta$ for $\theta \in [0, \frac{\pi}{2}]$.

Answers

1. $\ln(2-x) = \sum\limits_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \ln 2 + \sum\limits_{n=1}^{\infty} \frac{-x^n}{2^n n}$: $(R = 2)$

2. $\cos(\pi x) = \sum\limits_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} = \sum\limits_{n=0}^{\infty} \frac{(-1)^{n+1}\pi^{2n}(x-1)^{2n}}{(2n)!}$

   $|R_n(x)| \leq \frac{\pi^{n+1}|x-1|^{n+1}}{(n+1)!} \to 0$ for all x

3. $e^x \cosh 2x = (1 + x + \frac{x^2}{2!} + \cdots\ldots)(1 + \frac{(2x)^2}{2!} + \cdots) = 1 + x + \frac{5}{2}x^2 + \frac{13}{6}x^3 + \cdots$

4. $\frac{x^2}{\cos x - 1} = \frac{x^2}{-\frac{x^2}{2!} + \frac{x^4}{4!} - \cdots} = -2 - \frac{x^2}{6} - \frac{x^4}{120} + \cdots$

5. $\frac{1}{\sqrt[3]{1+x}} = 1 - \frac{x}{3} + \frac{2x^2}{9} - \frac{14x^3}{81} + \cdots$
   Thus, $\frac{1}{\sqrt[3]{1.1}} \approx 1 - \frac{1}{30} + \frac{1}{450}$. Since the series is alternating the error for this sum is less than the size of the next term, which is $\frac{7}{40500}$, which is less than 0.001.

6. Find the sum:
   (a) $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(\sqrt{3})^{2n+1}(2n+1)} = \tan^{-1}(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$
   (b) $\sum\limits_{n=0}^{\infty} \frac{3}{2^n n!} = 3\sqrt{e}$
   (c) $\sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = \frac{\sin x}{x}$
   (d) $\frac{4}{2!} + \frac{8}{3!} + \frac{16}{4!} + \frac{32}{5!} + \ldots. = e^2 - 3$

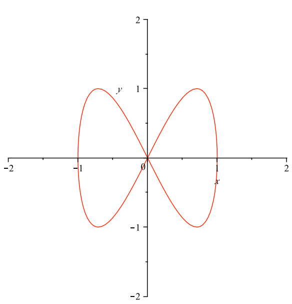
7. $T_3(x) = x + \frac{1}{6}x^3$

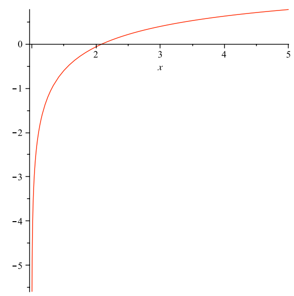
8. (a) $2 + \frac{x-8}{12} - \frac{(x-8)^2}{288}$: $|R_2| \leq \frac{f^{(3)}(7)\cdot 1^3}{3!} \approx 0.00034$
   (b) $x^2 - \frac{x^4}{6}$: $|R_4| \leq \frac{f^{(5)}(1)\cdot 1^5}{5!} \approx 0.0396$

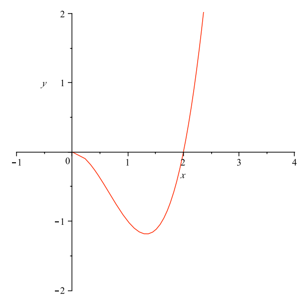
9. $-(x - \frac{\pi}{2}) + \frac{1}{6}(x - \frac{\pi}{2})^2$: $\cos 80^\circ = \cos\frac{4\pi}{9} \approx 0.174$

10. $T_2(x) = s(0) + s'(0)x + \frac{s''(0)}{2}x^2 = 20x + x^2$
    $T_2(1) = 21$ m: No

11. Prove by mathematical induction or directly consider the k^{th} derivative of the polynomial T_n.

12. 

13. 

14. 

15. $y = 2x\sqrt{1-x^2}$

16. $y = \ln(\ln(x)) - \ln(2) + 1$

17. $y = x^3 - 2x^2$

18. $x = \cos(t)$, $y = 2\sin(t)$

19. $x = t$, $y = t^2 + 2t - 1$

20. $y = \frac{3}{2}x - \frac{1}{2}$

21. vertical at $t = 1/2$, horizontal at ± 1, concave up when $t > 1/2$, concave down when $t < 1/2$.

22. $\frac{8\sqrt{2}}{15}$

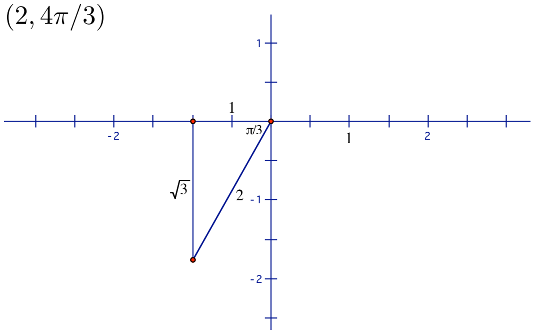
23. $\frac{5\pi}{2}$

24. $-\sqrt{10}/3 + \ln(3 + \sqrt{10}) + \sqrt{2} - \ln(1 + \sqrt{2})$

25. $e^3 - e^{-3}$

26. $\frac{2}{1215}\pi(247\sqrt{13} + 64)$.

$(2, 4\pi/3)$

27. 

28. Convert $(2, \pi/3)$ to Cartesian coordinates. $x = 2\cos(\pi/3) = 1$, $y = 2\sin(\pi/3) = \sqrt{3}$

Convert $(4, 2\pi/3)$ to Cartesian coordinates. $x = 4\cos(2\pi/3) = -2$, $y = 4\sin(2\pi/3) = 2\sqrt{3}$

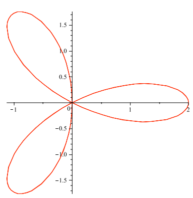
Find the distance between $(1, \sqrt{3})$ and $(-2, 2\sqrt{3})$ in Cartesian coordinates.

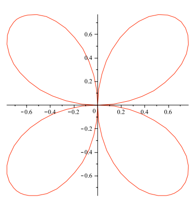
$\sqrt{9+3} = \sqrt{12} = 2\sqrt{3}$

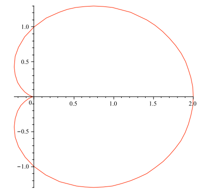
29. $r = 3$.

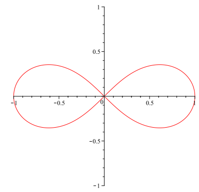
30. $r = 4\sin\theta$ gives $r^2 = 4r\sin\theta$. In Cartesian coordinates $x^2 + y^2 = 4y$ or $x^2 + y^2 - 4y + 4 = 4$ or $x^2 + (y-2)^2 = 4$

This is a circle of radius 2 centered at (0,2).

31. 

32. 

33. 

34. 

35. $r = 6\sin\theta$ and $r = 2 + 2\sin\theta$ intersect at $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$. Use symmetry to get the integral:

$$A = 2\int_{\pi/6}^{\pi/2} \frac{1}{2}\Big((6\sin\theta)^2 - (2+2\sin\theta)^2\Big)\, d\theta = 4\pi$$

36. One petal is drawn for $\theta \in [-\frac{\pi}{6}, \frac{\pi}{6}]$. Use symmetry to get the integral:

$$A = 2\int_0^{\pi/6} \frac{1}{2}\cos^2 3\theta \, d\theta = \frac{\pi}{12}$$

37. The length of the curve is the integral:

$$L = \int_0^{\pi} \sqrt{\theta^2 + 1}\, d\theta = \frac{1}{2}\pi\sqrt{\pi^2+1} + \frac{1}{2}\ln\left(\pi + \sqrt{\pi^2+1}\right)$$

38. The length of the curve is the integral:

$$L = \int_0^{\pi/2} \sqrt{(\theta + \sin\theta)^2 + (1+\cos\theta)^2}\, d\theta$$