## Math 113 Exam 2 Practice

September 25, 2014

Exam 2 will cover 7.4, 7.5, 7.7, 7.8, 8.1-3 and 8.5. Please note that integration skills learned in earlier sections will still be needed for the material in 7.5, 7.8 and chapter 8. This sheet has three sections. The first section will remind you about techniques and formulas that you should know. The second gives a number of practice questions for you to work on. The third section give the answers of the questions in section 2.

## Review

## 7.4: Integration of Rational Functions by Partial Fractions

Rational functions consist of fractions of polynomials. We can split rational functions into simpler pieces by partial fractions. Remember that partial fraction decompositions are based on linear and quadratic factors in the denominator. For each linear factor, we have a term with a constant in the numerator and the factor in the denominator. For each irreducible quadratic, we have a term with a linear function in the numerator and the quadratic in the denominator. For example,

$$\frac{x^2 - x + 2}{(x-1)(x+2)(x^2 + x + 3)} = \frac{A}{x-1} + \frac{B}{(x+2)} + \frac{Cx+D}{x^2 + x + 3}.$$
(1)

We just need to determine the values of A, B, C and D. This is done by plugging in values for x: You need to plug in as many numbers as you have constants. Using some numbers, (like -2 and 1 in this case) makes your life easier, but any four numbers will do. Notice that if we multiply equation 1 by the denominator on the left side, we get

$$x^{2} - x + 2 = A(x+2)(x^{2} + x + 3) + B(x-1)(x^{2} + x + 3) + (Cx+D)(x-1)(x+2).$$
(2)

Letting x = -2 in equation 2 gives 8 = -15B, so B = -8/15. Letting x = 1 gives 2 = 15A, so A = 2/15. Letting x = 0 gives 2 = 5A - 3B - 2D. Knowing A and B helps us to find D. Finally, if x = -1, 4 = 9A - 6B + 2C - 2D, allowing us to solve for C.

Remember that repeated factors must give repeated terms with increasing exponent in the denominator. For example,

$$\frac{x^3 + 2x^2 + 2x - 5}{(x-2)^3(x^2+9)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2}$$

Finally, remember partial fractions only works if the degree in the numerator is less than the degree in the denominator. Otherwise, you need to divide and use partial fractions on the remainder.

#### 7.5 Strategy for Integration

You may have noticed that we have really only used two techniques of integration in chapter 7: substitution and integration by parts. Everything else is just manipulation of those two techniques. Luckily, most integrals that require a specific technique have patterns that help remind us. Those patterns have been discussed earlier. You are expected to know them.

A general strategy for attacking unknown integrals is as follows:

- 1. Can substitution be used to simplify the integral? If so, do this first. (Note that in this section, substitution is often just the first step. After the substitution, some other technique needs to be applied.)
- 2. If you cannot use substitution, or you have used it but the integral is still not simple enough, look for patterns.
  - (a) If the integral is a product of trig functions, use the patterns we learned in 7.2 to tackle it.
  - (b) If the integral has a sum or difference of squares in the integrand, use trig substitution.
  - (c) If the integrand is a rational function, use partial fractions.
- 3. If you cannot see one of the above patterns, and cannot use substitution, try integration by parts.

## 7.7 Approximate Integration

In this section we concentrate on two things: How can we efficiently approximate the solution to a definite integral, and how can we approximate the error.

You are expected to know how to calculate the following approximations of a definite integral  $\int_a^b f(x) dx$ :

(a) the left and right hand sum

Left Hand Sum: 
$$L_f = \Delta x \sum_{k=1}^n f(x_{k-1})$$
  
Right Hand Sum:  $R_f = \Delta x \sum_{k=1}^n f(x_k)$ 

(b) the Trapezoid rule

$$T_f = \frac{\Delta x}{2} \left( f(a) + f(b) + 2 \sum_{k=1}^{n-1} f(x_k) \right)$$

(c) the Midpoint rule

$$M_f = \Delta x \sum_{k=1}^n f\left(\frac{x_k + x_{k-1}}{2}\right)$$

(d) Simpson's rule

$$S_f = \frac{\Delta x}{3} \left[ f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \dots + 4f(x_{n-1}) + f(x_n) \right].$$

Remember, n must be even to use Simpson's rule.

It is unlikely that you will need to calculate the left or right hand sums, but you will still need to know about them.

You are expected to know how each of these rules behave based on standard properties of the function (monotonicity, concavity, etc). For example, it is well known that the left hand sum is a lower bound and the right hand sum is an upper bound of the definite integral of increasing functions. What behavior is true for Midpoint and trapezoid?

You are expected to know the error estimates of Trapezoid, Midpoint and Simpsons:

#### Trapezoid

$$|E_T| \le \frac{K(b-a)^3}{12n^2}$$

with  $|f''(x)| \leq K$  on [a, b].

#### Midpoint

$$|E_M| \le \frac{K(b-a)^3}{24n^2}$$

with  $|f''(x)| \leq K$  on [a, b].

### Simpson's

$$|E_S| \le \frac{K(b-a)^5}{180n^4}$$

with 
$$|f^{(4)}(x)| \le K$$
 on  $[a, b]$ .

You can be expected to use the error estimates in one of two ways: to estimate the error of the calculation for a particular value of n, or to find a value for n that gives an error no more than some stated value.

### 7.8 Improper Integrals

Remember, there are two types of improper integral:

• Infinite length: Integrals of the type

$$\int_{-\infty}^{a} f(x) \, dx, \quad \int_{a}^{\infty} f(x) \, dx.$$

• Unbounded integrand. Integrals of the type

$$\int_{a}^{b} f(x) \, dx, \quad \int_{c}^{a} f(x) \, dx$$

where f has an infinite discontinuity at a.

Remember, in each type, there is a "problem" that a definite integral cannot handle. We remove the problem by turning it into a limit:

$$\int_{a}^{\infty} f(x) dx = \lim_{b \to \infty} \int_{a}^{b} f(x) dx$$
$$\int_{c}^{a} f(x) dx = \lim_{b \to a^{-}} \int_{c}^{b} f(x) dx.$$

Some things to remember when calculating improper integrals:

- Do not forget to set up an improper integral as a limit. You will likely have points deducted if you do not. It is the only way for the grader to tell that you know what you are doing.
- Watch out for infinite discontinuities in the middle of the interval. You must split the integral at the discontinuity in that case.

#### 8.1 Arc Length

To find the length of a curve given by f(x),  $a \le x \le b$ , the formula is

$$s = \int_{a}^{b} \sqrt{1 + (f'(x))^2} \, dx.$$

Notice that there is a reason that this section is not in chapter 6. Arc length integrals can be difficult to solve. You may need any of the techniques from Chapter 7 to calculate the arc length.

If the integral is complicated, you may wish to simplify it first. for example, when f' is a fraction, you may wish to rewrite  $1 + (f'(x))^2$  as a single fraction before you proceed.

## 8.2 Surface Area

Here are the important steps to keep in mind when solving this problem:

- First sketch the curve and identify the axis of rotation. Choose a variable, either x or y, to serve as our independent variable. If the curve is given as a function of x (e.g.,  $y = x^2 + 1$ ), then we will want to choose x as our independent variable, while if the curve is given as function of y (e.g.,  $x = y^3 y + 1$ ), we will want to choose y as our independent variable. This is the variable with respect to which we will be integrating. If the curve is given implicitly (e.g.,  $x^2 + y^2 = 1$ ), then we may choose either variable and solve for the other variable.
- Next consider the circumferences of the circles formed by revolving points on the curve about the axis of rotation. The circumference of such a circle will be  $2\pi r$ , where r is the distance to the axis of rotation. This circumference should be expressed as a function of either x or y, whichever one we chose in the previous step to be our independent variable.
- The general formula for the area of a surface of revolution of this type is

$$S = \int_{a}^{b} (\text{circumference}) \, ds$$

where circumference is what we found above, and ds is either  $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$  or  $\sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$ , according to whether x or y is our independent variable, respectively.

Examples:

Rotating f(x),  $a \le x \le b$  about the x axis:  $\int_{a}^{b} 2\pi f(x)\sqrt{1 + (f'(x))^2} \, dx$ . Rotating f(x),  $a \le x \le b$  about the y axis:  $\int_{a}^{b} 2\pi x\sqrt{1 + (f'(x))^2} \, dx$ . Rotating f(x),  $a \le x \le b$  about the axis y = c:  $\int_{a}^{b} 2\pi |f(x) - c|\sqrt{1 + (f'(x))^2} \, dx$ . Rotating f(x),  $a \le x \le b$  about the axis x = d:  $\int_{a}^{b} 2\pi |x - d|\sqrt{1 + (f'(x))^2} \, dx$ .

Exercise: Write the appropriate formulas for a function of y.

Note that you may need the same techniques to solve these integrals as you do in section 8.1.

## 8.3 Applications to Physics and Engineering

#### Fluid pressure

The calculation of fluid pressure on a plate vertically suspended in a liquid is

$$\int_{a}^{b} \omega D(h) L(h) \, dh$$

where D(h) is the depth at h, and L(h) is the width of the plate at h.

#### Centroid

There are three types of problems that you will need to be able to calculate.

- Using the property of sums of moments to be able to find centroids and center of mass. (Used in point masses, and in regions where the area is easily calculated.)
- Finding the moments with respect to the x and y axes, and the centroid of a region bounded by two functions. For example, if the region is bounded by

$$f(x) \le y \le g(x), \quad a \le x \le b,$$

then we have:

Moment about the y axis:
$$M_y = \int_a^b x(f(x) - g(x)) dx$$
  
Moment about the x axis: $M_x = \int_a^b \frac{1}{2} (f^2(x) - g^2(x)) dx$ .

Since the area of the region is

$$A = \int_{a}^{b} (f(x) - g(x)) \, dx,$$

the centroid is given by

$$\overline{x} = \frac{M_y}{A}, \quad \overline{y} = \frac{M_x}{A}.$$

• finding the center of mass of a plate. Here, we have a density. (mass is density times volume). If the density is constant, the moments are just the calculations above times the density. The mass of the object is also the area times the density. The center of mass, however, is the same as the calculation for the centroid. In the case of a density that is not constant, the situation is more complicated. However, since the homework only has constant density, that is all that need concern us for this exam.

You may need to rework the above equations for regions contained by functions of y.

#### Geometric properties of the centroid

We now turn to some properties of the centroid which can simplify certain geometrical problems.

- 1. Centroids of triangles are given geometrically If you are trying to find the centroid of a triangle, you can do the following: Measure 2/3 along the line connecting a vertex to the midpoint of the opposite side. That's the centroid! (If you draw the lines from each vertex to the midpoint of the opposite side, they all intersect in the centroid.)
- 2. Use symmetry The centroid will always lie on a line of symmetry (if the object possesses one). For example, suppose we have a triangle with vertices at (0,0), (2,0) and (1,5). This is an isosceles triangle. By the symmetry, it is clear that  $\bar{x} = 1$ . From the last item, since the line from the vertex above to the midpoint is vertical, we see that  $\bar{y} = 5/3$ .
- 3. Moments Add! If you are trying to find the moment of a complicated region, you can split the region into simpler regions and add the moments together. For example, suppose a circle of radius 1 and a square of length 1 are placed side by side. Where is the centroid of the system?



The centroid of the circle is (-1, 1). Thus, the moment of the circle about the y axis is -1 times the area of the circle:  $-\pi$ . Similarly, the moment of the circle about the x axis is  $\pi$ . Since the centroid of the square is at (1/2, 1/2), the moments of the square about the y and x axes are both 1/2 (since the area of the square is 1). Thus, the total moment about the y axis is  $-\pi + 1/2$ . Also the total moment about the x axis is  $\pi + 1/2$ . Hence, the centroid is given by

$$\overline{x} = \frac{-\pi + 1/2}{\pi + 1}$$
$$\overline{y} = \frac{\pi + 1/2}{\pi + 1}.$$

and

4. Other calculations look like moment calculations Recall that when we developed the calculations for force due to fluid pressure, we took a small strip and multiplied the area by  $\omega \times \text{depth}$ . That is like calculating a moment! It should be no surprise, therefore, that if the force due to fluid pressure on the (submerged, vertical) plate is

(weight density of liquid)  $\times$  (depth of centroid)  $\times$  (area of plate).

We also have the first theorem of Pappus for the calculation of volumes of rotation:

Volume =  $2\pi \times (\text{distance from centroid to rotation axis}) \times (\text{area}).$ 

You should only use these formulas either when you already have the centroid, or when it is easy to find.

For example, suppose we wish to find the force on the side of a circular plate of radius 2 feet that is submerged in water to a point 5 feet from the top of the plate. Since the centroid of the plate is 7 feet from the surface, the force is given by

 $F = 62.5 \times 7 \times 4\pi = 437.5\pi lb.$ 

Notice how much easier the calculation becomes! That is because the centroid was easy to find.

#### 8.5 Probability

This section is concerned with the probability of a continuous random variable. A continuous random variable X is described by a **probability density function** f(x) with the following properties:

a)  $f(x) \ge 0$  on  $(-\infty, \infty)$ .

b) 
$$\int_{-\infty}^{\infty} f(x) dx = 1.$$

With a probability density function, you can perform the following calculations:

- a) Find the probability that X is between A and B:  $P(A < X < B) = \int_{A}^{B} f(x) dx$
- b) Calculate the mean:  $\mu = \int_{-\infty}^{\infty} x f(x) dx$
- c) Calculate the median: Find m so that  $\int_m^\infty f(x)\,dx=\frac{1}{2}$
- d) Calculate the standard deviation:  $\sigma = \sqrt{\int_{-\infty}^{\infty} (x-\mu)^2 f(x) \, dx}$

## Questions

Try to study the review notes and memorize any relevant equations **before** trying to work these equations. If you cannot solve a problem without the book or notes, you will not be able to solve that problem on the exam.

- 1.  $\int \frac{6}{x^3 + 2x^2 + x} dx$
- 2.  $\int \frac{x^2 + x 5}{x^2 1} dx$
- 3.  $\int \frac{x-1}{x^3+x} dx$
- 4.  $\int \frac{x^3}{x^4 + 2x^2 + 1} dx$
- 5.  $\int_{9}^{16} \frac{\sqrt{x}}{x-4} dx$  [Hint: Use a rationalizing substitution]
- 6.  $\int_0^{\frac{\pi}{2}} \frac{1}{2-\cos x} dx$  [Hint: Use the substitution  $t = \tan(\frac{x}{2})$ ]
- 7. Estimate  $\int_0^4 e^{-x^2} dx$  with midpoint, trapezoid, and simpson for n=4 and 8.
- 8. Calculate a bound on the error for the trapezoid calculation in the last question (n=4 only).
- 9. How large does n need to be in order for midpoint to have an error no larger than  $10^{-5}$ ?
- 10. How large does n need to be in order for Simpson's to have an error no larger than  $10^{-5}$ ?

For questions 11 to 17, evaluate the integral, or show that it diverges.

11. 
$$\int_{2}^{\infty} \frac{1}{x \ln x} dx$$
  
12. 
$$\int_{1}^{\infty} \frac{\ln x}{x^{4}} dx$$

13. 
$$\int_{1}^{\infty} \frac{1}{(2x+1)^3} dx$$

- 14.  $\int_0^1 \ln x \, dx$
- 15.  $\int_{\pi/2}^{\pi} \sec x \, dx$
- 16.  $\int_1^\infty \ln x \ dx$
- 17.  $\int_1^4 \frac{dx}{x^2-4}$

For questions 18 to 19, use the Comparison Theorem to show whether the improper integral converges or diverges.

18. 
$$\int_{1}^{\infty} \frac{\tan^{-1} x}{x^{2}} dx$$
  
19. 
$$\int_{1}^{\infty} \frac{2 + e^{-x}}{\sqrt[3]{x^{2}}} dx$$

- 20. Find the arc length function for the curve  $y = 2\sqrt{x}$  with initial point (0,0).
- 21. Find the arc length function for the curve  $y = 2x^{\frac{3}{2}}$  with initial point (0,0).
- 22. Find the arc length of the curve C :  $y = 3 + \frac{1}{2}\cosh(2x)$  for  $0 \le x \le 1$ .

- 23. Find the arc length for the curve  $y = \sqrt{x x^2} + \sin^{-1}(\sqrt{x})$  for  $0 \le x \le 1$ . In questions (24) to (28), find the area of the surface formed by revolving the curve about the specified axis:
- 24. y = 2x + 1,  $1 \le x \le 3$ , about the y-axis.
- 25.  $x = \sin y, 0 \le y \le \pi$ , about the *y*-axis.
- 26.  $y = 1 + x^2, -1 \le x \le 1$ , about the line y = 2.
- 27.  $y = x^3 + 1, 0 \le x \le 1$ , about the line y = 1.

28. 
$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$
, about the *x*-axis

- 29. A circular pipe of 4m diameter has a plate that regulates the flow. When the plate is in position, the flow is completely cut off. What is the force on the plate in this position, if the pipe is full of water on one side?
- 30. Find the force due to fluid pressure on one side of a plate in the shape of the upper half of a regular hexagon of diameter 8 feet if the hexagon is sitting vertically in water with the (flat) top of the hexagon at a depth of 15 feet.
- 31. A cylindrical barrel with height 2m and diameter 1m is filled to the brim with oil. If the weight density of oil is 900 kilograms per cubic meters, what is the force due to fluid pressure on the side of the barrel?
- 32. The masses  $m_i$  are located at the points  $P_i$ . Find the total moments  $M_y$  and  $M_x$  of the system, and find the center of mass of the system.  $m_1 = 4, m_2 = 2, m_3 = 3$  $P_1 = (1, 2), P_2 = (-2, 3), P_3 = (4, -1)$
- 33. Let R be the region bounded by  $f(x) = \sin(x)$  and  $g(x) = \cos(x)$  for  $x \in [\pi/4, 5\pi/4]$ . Sketch the region R, and find the centroid (center of mass) of R.
- 34. Let R be the region bounded by  $f(x) = e^x$  and  $g(x) = x^2$  for  $x \in [0, 1]$ . Sketch the region R, and find the centroid (center of mass) of R.
- 35. Let R be the region bounded by the curves  $f(x) = 2 x^2$  and  $g(x) = x^2$ . Sketch the region R, and find the centroid (center of mass) of R.
- 36. Let R be the region consisting of two adjacent circular discs described by the equations  $(x-1)^2 + y^2 = 1$  and  $(x-3)^2 + y^2 = 1$ . Use the theorem of Pappus to find the volume of the solid obtained by rotating R about the y-axis.
- 37. Let R be the cross shape formed by taking a square of side length 4, and cutting out of each corner a smaller square of side length 1. Use the theorem of Pappus to find the volume of the solid obtained by rotating R about one of its original sides.

- 38. Find the force due to fluid pressure on a vertical plate in the shape of an equilateral triangle (2 feet on a side), with the lower side parallel to the surface, and the upper vertex at a depth of 15 feet.
- 39. Find the centroid of the following object: (each tic mark is one unit)

ark is one unit)

For questions 40 to 42, a) determine the value for c that makes f(x) a probability density function, b) find P(A < X < B) and P(X > B), c) find  $\mu$ , and d) find  $\sigma$ .

- 40.  $f(x) = cxe^{-2x}$  on  $[0, \infty)$ , A = 0, B = 1.
- 41.  $f(x) = \frac{c}{(1+x^2)^3}$  on  $(-\infty, \infty)$ , A = -1, B = 2.
- 42. f(x) = cx(1-x) on [0,1], A = 0, B = 1/2.

# Answers

1. 
$$6 \ln |x| - 6 \ln |x + 1| + \frac{6}{x+1} + C$$
20.  $\sqrt{x + x^2} + \ln (\sqrt{x} + \sqrt{1 + x})$ 2.  $x - \frac{3}{2} \ln |x - 1| + \frac{5}{2} \ln |x + 1| + C$ 21.  $L(x) = \frac{2(1 + 9x)^{\frac{3}{2}} - 2}{27}$ 3.  $-\ln |x| + \frac{1}{2} \ln |x^2 + 1| + \tan^{-1} x + C$ 22.  $\frac{1}{2} \sinh 2$ 4.  $\frac{1}{2} (\ln |x^2 + 1| + \frac{1}{x^2 + 1}) + C$ 23. 24.  $\frac{1}{2} (\ln |x^2 + 1| + \frac{1}{x^2 + 1}) + C$ 24.  $8\pi\sqrt{5}$ 5.  $2(\ln 5 - \ln 3 + 1)$ 25.  $2\pi(\sqrt{2} + \ln(\sqrt{2} + 1))$ 6.  $\frac{2\sqrt{3}}{9}\pi$ 27.  $\frac{\pi}{27}(10\sqrt{10} - 1)$ 7.  $n = 4$ : Midpoint .886135246928.  $\pi \left(18 + \frac{24}{\sqrt{5}} \ln \left(\frac{6 + \sqrt{5}}{2}\right)\right)$ Simpson .836214264629.  $8\pi\omega = 246300N$  $n = 8$ : Midpoint .886226918330.  $\omega(180\sqrt{3} + 40) = 21985.57$  lbs.Trapezoid .88623636831.  $2\pi\omega = 2\pi \cdot 9.8 \cdot 900 = 55417.69$  N8.  $|f''(x)| < 2$  on  $[0, 4]$ .  $|E| < \frac{2}{3}$ .32.  $M_y = 12, M_x = 11, (\bar{x}, \bar{y}) = (4/3, 11/9)$ 9. 73131.  $\left(\frac{3\pi}{4}, 0\right)$ 10.  $|f^{(4)}(x)| < 12. n = 52.$ 34.  $\left(\frac{9}{12e - 16}, \frac{3(5e^2 - 7)}{20(3e - 4)}\right)$ 12.  $\frac{1}{9}$ 35.  $(0, 1)$ 13.  $\frac{1}{36}$ 36.  $8\pi^2$ 14. -137.  $48\pi$ 15. Diverges.38.  $62.5(15 + \frac{2}{3}\sqrt{3})\sqrt{3} = 1748.80b$ 16.  $\infty$ 39.  $(\frac{-211 + 64\pi}{98 + 32\pi}, \frac{112 + 64\pi}{98 + 32\pi}) = (-0.05, 1.58)$ 18. Converges. Make the comparison  $\frac{4n^{-1}x}{x^2} < \frac{\pi/2}{x^2}$ .40.  $c = 4, 1 - 3e^{-2}, 3e^{-2}, 1, 1/\sqrt{2}$ 41.  $c = 8/(3\pi), 0.95891, 0.0033104, 0, 1/\sqrt{6}$ 42.  $c = 6, 1/2, 1/2, 1/2, 1/2, \sqrt{5}$ .