Math 113 Exam 3 Practice

Exam 3 will cover 11.1-11.9. This sheet has three sections. The first section will remind you about techniques and formulas that you should know. The second gives a number of practice questions for you to work on. The third section give the answers of the questions in section 2.

Review

Sequences

Sequences are an important part of Chapter 11, because so much of what we do involves them. You need to be able to take a sequence and determine its behavior. Is is increasing or decreasing? Does it converge in the limit? What does it converge to? Some theorems may be of help here:

1. If a sequence converges, it is bounded.
2. If a sequence is bounded and is (eventually) increasing or decreasing, then it converges.
3. If a sequence $\{a_n\}$ matches a function f (i.e. $f(n) = a_n$) and
$$\lim_{n\to\infty} f(x) = L,$$
then the limit of the sequence is also L.

Rule 3 is useful because we can use everything we know about limits of functions to find limits of sequences. Since L'Hopital's rule is one of them, you should expect to use it. There are other rules about sums of sequences and products of sequences, etc. You are advised to review them in the text.

Important Sequences

Some limits occur often enough that it is advisable to know about them in advance. For example, Dr. McKay expects all of his students to know the following:

1. If c is a real, positive number, then $c^{1/n} \to 1$.
2. If c is a real, positive number, then $\dfrac{1}{n^c} \to 0$.
3. $\dfrac{c^n}{n!} \to 0$.
4. $n^{1/n} \to 1$.
5. $\left(1 + \dfrac{c}{n}\right)^n \to e^c$.

All of the above limits except the third can be proven using L'Hopital's rule. The third is a bit tricky but can be done by noticing that the $(n+1)$st term of the series is $\dfrac{c}{n+1}$ times the nth term.

If you encounter these limits in a problem, you are welcome to use what you know about them and move on.

Recursive Sequences

Most sequences that we deal with have a rule we can apply to find the nth term of the sequence. **Recursive sequences**, however, only have a rule that allows us to find the nth term if we know all of the other terms that come before it. The most famous recursive sequence is the Fibbonaci sequence given by

$$a_1 = 1,\ a_2 = 1,\ a_n = a_{n-1} + a_{n-2}.$$

Generally, it is difficult to tell what a recursive sequence does. The theorems listed above can show that a recursive sequence converges. For example if the sequence can be shown to be bounded and is increasing, then it must have a limit. If a recursive sequence converges, then finding the limit of the sequence is easy: Let L represent the limit. Since the sequence converges to L, every sequence element in the recursion formula converges to L also. For example, suppose we wish to find the limit of

$$a_1 = 1, a_2 = 1, a_n = \frac{1}{2}a_{n-1} - \frac{1}{2^{n-1}}.$$

If we are reasonably sure the limit exists, we can replace a_n by L to get

$$L = \frac{1}{2}L - 0$$

You can then use this to solve for L. Make sure the series converges however, or you may arrive at the wrong conclusion. For example, suppose we have the recursive sequence

$$a_1 = 2, a_n = 2a_{n-1} - 1.$$

Using this technique gives $L = 2L - 1$, which yields $L = 1$. However, it is not hard to see that this sequence diverges.

Series

In this section we learned about convergent and divergent series. A series converges if the sequence of partial sums converge. There are some particular types of series that we learned about:

Geometric Series $\sum_{n=0}^{\infty} ar^n$. We learned that the geometric series converges to $\frac{a}{1-r}$ if $|r| < 1$ and diverges otherwise. We saw several applications where we could write a problem in terms of a geometric series.

Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$. We saw by examination of the partial sums s_{2^n} that this diverges. The integral test also shows the divergence of this series.

Alternating Harmonic Series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ We know by a later test that the alternating Harmonic series converges. We know by the Maclaurin series of $\ln(1+x)$ that it converges to $\ln(2)$.

Telescoping Series This is a series where the partial sum collapses to the sum of a few terms. We can then take the limit of the partial sum to see what the series converges to.

Note that in 11.2 the **only** series whose sums we could calculate were geometric and telescoping.

Tests for Convergence

We learned about the following tests for convergence:

Divergence Test If $a_n \not\to 0$ then $\sum a_n$ diverges. This is an excellent test to start with because the limit is often easy to calculate. Keep in mind, however, if the limit **is** 0, then the Divergence test tells you nothing. You must try some other test.

p series If you recognize a series as a p series,

$$\sum \frac{1}{n^p}$$

then you can use the fact that a p series converges when (and only when) $p > 1$.

Geometric series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ but only when $|r| < 1$.

Comparison Test To use the comparison test, we need to have a large group of test series available. We also need to know if these test series converge or not. The most common test series for the comparison test are the p series and the geometric series. If the series "acts like" a p series, or "acts like" a geometric series, then you may wish to use the comparison test. Remember, if $0 \leq a_n \leq b_n$ and

- $\sum b_n$ converges, then $\sum a_n$ converges.
- $\sum a_n$ diverges, then $\sum b_n$ diverges.

Limit Comparison Test This test works well for the type of problems that also work with the comparison test, but is somewhat easier. You still need the test series, but you don't need to work to make the terms of the series greater than or less than some known series. You only need to check the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n}.$$

If it is finite and positive, then both series converge or both diverge. Since you already know about one of them, you then know about the other.

Integral Test If we are trying to determine whether $\sum a_n$ converges, and there is a function $f(x)$ with $f(n) = a_n$, then the sum converges iff

$$\int_a^\infty f(x)\,dx \text{ converges.}$$

(We assume that both the series $\{a_n\}$ and $f(x)$ are positive.) So the integral test is handy if the associated function can be integrated without too much difficulty.

Alternating Series Test To use the alternating series test, you need to verify three things: The series is alternating. (This can usually be done by inspection). The terms of the series converge to 0. (Hopefully you did this when you applied the Divergence test.) Finally, the terms of the absolute values are decreasing. The second statement does not necessarily imply the third. If this is true, then the alternating series test tells us the series converges.

Ratio Test If

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L,$$

then the series is absolutely convergent if $L < 1$ and divergent if $L > 1$. If $L = 1$, the test fails. This test works really well when a factorial is present in a_n.

Root Test If

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L,$$

then the series is absolutely convergent if $L < 1$ and divergent if $L > 1$. If $L = 1$, the test fails. This test works really well when there are powers of n in a_n.

Remember, the Integral test and the comparison tests only work when the series has non-negative terms. If you have a series where the terms are both positive and negative, then you must be able to say whether the series converges absolutely, converges conditionally, or diverges. It **is** one of these. These are mutually exclusive conditions.

Estimating the tail

In an infinite series, **the tail** is a term usually used to indicate the "last" part of the series. For example, if we wish to approximate the sum of the following convergent series,

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^3},$$

then we can write it as

$$\sum_{n=0}^{k} \frac{1}{(2n+1)^3} + \sum_{n=k+1}^{\infty} \frac{1}{(2n+1)^3}$$

The part that is still an infinite sum is called the tail. The sum of the tail is called the **error** of our approximation. If we can test convergence of a series by the integral test, then there is an easy way to find an estimate of the tail: Assume $f(x)$ is defined on $[b, \infty)$ for some b, and $f(n) = a_n$. Then

$$\int_{k+1}^{\infty} f(x)\,dx \le \sum_{n=k+1}^{\infty} a_n \le \int_k^{\infty} f(x)\,dx.$$

For example suppose that we sum the first 5 terms of the above series:

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \approx \sum_{n=0}^{4} \frac{1}{(2n+1)^3}$$

$$= 1 + \frac{1}{27} + \frac{1}{125} + \frac{1}{343} + \frac{1}{729} = 1.049324231$$

How close is this? We find that

$$\int_5^{\infty} \frac{1}{(2x+1)^3}\,dx = \frac{1}{484} = 0.002066115702,$$

and

$$\int_4^{\infty} \frac{1}{(2x+1)^3}\,dx = \frac{1}{324} = 0.003086419753.$$

Thus, the error is between these two numbers.

If we can use the comparison test to find convergence, then we can sometimes still use the above formula, but only for upper bounds. For example, if I am trying to estimate

$$\sum_{n=k+1}^{\infty} \frac{n-1}{n^3+1},$$

the fact that

$$\frac{n-1}{n^3+1} < \frac{1}{n^2}$$

means that

$$\sum_{n=k+1}^{\infty} \frac{n-1}{n^3+1} < \int_k^{\infty} \frac{1}{x^2}\,dx.$$

If our sum is alternating, with the absolute values decreasing to 0, then the tail is bounded by the absolute value of its first term. That is, if $a_n > 0$ for all n,

$$\left| \sum_{n=k+1}^{\infty} (-1)^n a_n \right| \leq a_{k+1}.$$

(Note that it does not matter if there is $(-1)^n$ or $(-1)^{n+1}$ in the sum. All that matters is that the sum is alternating, and satisfies the alternating series test.)

Power Series

Recall that a power series is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n.$$

The value c is called the **center** of the power series, and the values a_n are called the coefficients.

A power series is a way to represent a function. However, the power series may have a different domain than the function does. To find the domain of the power series, (called the **interval of convergence**), we do the following:

1. Apply the ratio or root test to the power series. If the limit is 0, the power series converges everywhere and the radius of convergence is ∞. If the limit is ∞, the power series converges only at the center, and the radius of convergence is 0. Otherwise, set the limit to be less than 1, and rework the inequality so it says $|x-c| < R$. R is the radius of convergence.

2. The power series is now guaranteed to converge absolutely on $(c-R, c+R)$, and diverge on $(-\infty, c-R) \cup (c+R, \infty)$. We now test the power series at the endpoints. Plug the endpoints $c-R$ and $c+R$ into the power series and use one of the **other** 5 tests (not Ratio, not Root) to determine whether they converge. State the interval of convergence using parentheses to indicate the power series does not converge at an endpoint, and a bracket to indicate it does.

Finding sums of series

Finding a power series that represents a specific function is the next topic. The first one we learned was the geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ iff } x \in (-1, 1).$$

We then found the sum of several series by differentiating, integrating, multiplying by x, etc.

In addition to the geometric series, we found the following power series. You should know them and their interval of convergence.

- $\tan^{-1} x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$, $[-1, 1]$
- $\ln(1+x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$, $(-1, 1]$
- $\cos x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, $(-\infty, \infty)$

Questions

Try to study the review notes and memorize any relevant equations **before** trying to work these equations. If you cannot solve a problem without the book or notes, you will not be able to solve that problem on the exam.

Determine whether each sequence in 1 to 4 is convergent. State what it converges to, if applicable. Is the sequence increasing or decreasing? Is the sequence bounded?

1. $a_n = \dfrac{9^{n+1}}{10^n}$
2. $a_n = \cos(n\pi/2)$
3. $a_n = \dfrac{n \sin n}{n^2+1}$
4. $a_n = \dfrac{n^3}{1+n^2}$.
5. Find the value that the sequence given by $a_1 = 1$, $a_n = \frac{1}{a_{n-1}+1}$ converges to.

   Determine whether the series is convergent or divergent. If it is convergent, find its sum.

6. $\displaystyle\sum_{n=2}^{\infty} \frac{k^2}{k^2-1}$
7. $\displaystyle\sum_{n=2}^{\infty} \frac{2}{k^2-1}$
8. $\displaystyle\sum_{n=0}^{\infty} \frac{3}{5^n}$

   For problems 9 to 17, determine whether the series converges or diverges. State the test you used.

9. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n}$
10. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^2$
11. $\displaystyle\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{1+k^2}$
12. $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{(k^3+2)^2}$
13. $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \ln k}$
14. $\sum_{n=1}^{\infty} \frac{n+3^n}{6^n}$
15. $\sum_{n=0}^{\infty} \frac{n^4+7n+8}{n^5+7n^4+13n^2+19n+23}$
16. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$
17. $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2+1}$
18. Show that $\frac{2}{5}$ is an upper bound on the error of $\sum_{n=1}^{\infty} \frac{1}{n^4+7}$ if the sum is approximated by the first two terms.
19. Approximate the sum of

   $$\sum_{n=1}^{\infty} \frac{n+1}{n^3}$$

   by summing the first 10 terms. Find a bound on the error of your approximation.

   For problems 20 through 25, determine whether the series is absolutely convergent, conditionally convergent, or divergent.

20. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$
21. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$
22. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{2n+1}$
23. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}$
24. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (\ln n)^2}{n}$
25. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^5+3}}$
26. Show that $\frac{1}{24}$ is an upper bound on the error of $\sum_{n=1}^{\infty} \frac{1}{n^4+7}$ if the sum is approximated by the first two terms.
27. Suppose the power series $\displaystyle\sum_{n=2}^{\infty} a_n (x+1)^n$ has a radius of convergence $R = 5$. List all possible intervals of convergence.
28. Find the radius and interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^n}$
29. Find the radius and interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(-4)^n (x-2)^n}{3+2n}$
30. Find the radius and interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{3^n}$
31. Find the radius and interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{n 2^n x^n}{n^3-1}$

32. Find the radius and interval of convergence of $\sum_{n=1}^{\infty} \frac{n!\,x^n}{100^n}$

33. Find a power series representation in powers of x for the function
$$f(x) = \frac{1}{3+x}$$
with interval of convergence.

34. Find a power series representation in powers of $(x-1)$ for the function $f(x) = \frac{1}{1+x}$ and give the interval of convergence.

35. Find a power series representation in powers of $(x-1)$ for $\ln(1+x)$.

36. What is the power series representation of $\frac{x^2}{(1-x)^2}$?

Answers

1. converges to 0, decreasing, bounded
2. diverges, not increasing or decreasing, bounded.
3. converges to 0, not increasing or decreasing, bounded.
4. diverges, increasing, bounded below.
5. $\dfrac{-1+\sqrt{5}}{2}$
6. diverges by the Divergence Test
7. Converges to 3/2 (Telescoping sum)
8. Converges to 15/4 (geometric series)
9. Use the integral test
$$\int_1^{\infty} x^2 e^{-x} dx = \frac{5}{e}$$
Therefore it converges by the integral test
10. Use the integral test
$$\int_1^{\infty} \left(\frac{\ln x}{x}\right)^2 dx = 2$$
Therefore it converges by the integral test.
11. Use the integral test
$$\int_1^{\infty} \frac{\tan^{-1} x}{1+x^2}\, dx = \frac{3\pi^2}{32}$$
Therefore it converges by the integral test
12. $$\int_1^{\infty} \frac{x^2}{(x^3+2)^2}\, dx = \frac{1}{9}$$
Therefore it converges by the integral test.
13. We have to be careful here since the function is not defined at $k=1$. By a change of variables, $k = n+1$ we see that $\sum_{k=2}^{\infty} \frac{1}{k \ln k} = \sum_{n=1}^{\infty} \frac{1}{(n+1)\ln(n+1)}$ and we can then use the integral test. The book notes that we can also simply change the limits of integration, though it does not state this as a theorem.
$$\int_1^{\infty} \frac{1}{(x+1)\ln(x+1)}\, dx = \infty - (-\infty)$$
Therefore it diverges by the integral test.
14. $\frac{n+3^n}{6^n} < \frac{3^n+3^n}{6^n} = 2(\frac{1}{2})^n$ and $\sum 2(\frac{1}{2})^n$ converges (geometric $r = \frac{1}{2}$). Thus $\sum_{n=1}^{\infty} \frac{n+3^n}{6^n}$ converges by Comparison Test.
15. $\lim_{n\to\infty} \frac{\frac{n^4+7n+8}{n^5+7n^4+13n^2+19n+23}}{\frac{1}{n}} = 1$ and $\sum \frac{1}{n}$ diverges. Hence $\sum_{n=0}^{\infty} \frac{n^4+7n+8}{n^5+7n^4+13n^2+19n+23}$ diverges by Limit Comparison Test.
16. $\frac{\ln n}{n} > \frac{1}{n}$. Thus $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ diverges by Comparison Test.
17. $\frac{\cos^2 n}{n^2+1} < \frac{1}{n^2+1} < \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges (p-series). Thus $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2+1}$ converges by Comparison Test.
18. Since $\frac{1}{n^4+7} < \frac{1}{n^4}$ it is sufficient to show that $\frac{2}{5}$ is a bound on the sum $\sum_{n=3}^{\infty} \frac{1}{n^4}$. Then $R_2 \leq \int_2^{\infty} \frac{1}{x^4} dx = \frac{1}{24}$.
19. $\sum_{n=1}^{10} \frac{n+1}{n^3} = 2.747299717$. Note that
$$\int_{10}^{\infty} \frac{x+1}{x^3}\, dx = .105, \quad \int_{11}^{\infty} \frac{x+1}{x^3}\, dx = 0.09504132231.$$
Thus, the sum lies in the interval $(2.842341039, 2.852299717)$.
20. Converges by the Alternating Series test. By the Integral Test, it does not converge absolutely. So it converges conditionally.
21. Diverges by the Test for Divergence.
22. Converges by the Alternating Series test. By the Limit Comparison test (with $b_n = \frac{1}{n}$), it does not converge absolutely. So it converges conditionally.
23. Converges absolutely by the Limit Comparison test (with $b_n = \frac{1}{n^2}$).

24. Converges by the Alternating Series test (Use L'Hôpital's rule). By the Integral Test, it does not converge absolutely. So it converges conditionally.

25. Converges absolutely by the Limit Comparison test (with $b_n = \frac{1}{n^{3/2}}$).

26. $R_2 = \sum_{n=3}^{\infty} \frac{1}{n^4+7} \leq \sum_{n=3}^{\infty} \frac{1}{n^4} \leq \int_2^{\infty} \frac{1}{x^4}\, dx = \frac{1}{24}$

27. (-6,4), (-6,4], [-6,4), [-6,4]

28. $\left|\frac{a_{n+1}}{a_n}\right| \to 0$ for all x: $R = \infty$, $I = (-\infty, \infty)$

29. $\left|\frac{a_{n+1}}{a_n}\right| \to 4|x-2| < 1$: $R = \frac{1}{4}$, $I = (\frac{7}{4}, \frac{9}{4}]$

30. $\left|\frac{a_{n+1}}{a_n}\right| \to \frac{|x|}{3} < 1$: $R = 3$, $I = (-3, 3)$

31. $\left|\frac{a_{n+1}}{a_n}\right| \to 2|x| < 1$: $R = \frac{1}{2}$, $I = [-\frac{1}{2}, \frac{1}{2}]$

32. $\left|\frac{a_{n+1}}{a_n}\right| \to \infty$ for all x: $R = 0$, $I = \{0\}$

33. $\frac{1}{3+x} = \frac{1}{3}\left(\frac{1}{1-(-\frac{x}{3})}\right) = \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n$
$= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^{n+1}}$ for $x \in (-3, 3)$

34. $\frac{1}{1+x} = \frac{1}{2+(x-1)} = \frac{1}{2} \cdot \frac{1}{1-\left(\frac{-(x-1)}{2}\right)}$
$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}(x-1)^n$ for $x \in (-1, 3)$

35. Integrate the previous solution to get
$\ln(1+x) = C + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n(n+1)}(x-1)^{n+1}$: $(C = \ln 2)$

36. $\frac{x^2}{(1-x)^2} = x^2 \frac{d}{dx}\left(\frac{1}{1-x}\right) = x^2 \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right)$
$= x^2 \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=1}^{\infty} n x^{n+1}$