Exam 3 Practice solutions

1. b), d)

2. b), f)

3. a)

4. (a) 3

   (b) $\begin{bmatrix} 1 & -2 & 2 \\ 2 & -4 & 4 \\ 3 & -6 & 6 \end{bmatrix}$

   (c) $\dfrac{\mathbf{y} \cdot \mathbf{u_j}}{\mathbf{u_j} \cdot \mathbf{u_j}}$

   (d) $\sqrt{15}$

   (e) λ is an eigenvalue and $\mathbf{x}$ is an eigenvector.

   (f) It is linearly independent.

5. Suppose that $\lambda = \alpha + \beta i$ is an eigenvalue of A and that $\mathbf{x} = \mathbf{u} + \mathbf{iv}$ is an associated eigenvector. Then,

$$A\mathbf{x} = \lambda\mathbf{x}.$$

However,

$$A\mathbf{x} = \mathbf{A}(\mathbf{u} + \mathbf{iv}) = \mathbf{Au} + \mathbf{iAv}.$$

Also,

$$\lambda\mathbf{x} = (\alpha + \beta\mathbf{i})(\mathbf{u} + \mathbf{iv}) = \alpha\mathbf{u} - \beta\mathbf{v} + \mathbf{i}(\beta\mathbf{u} + \alpha\mathbf{v}).$$

Hence

$$A\mathbf{u} = \alpha\mathbf{u} - \beta\mathbf{v},$$

while

$$A\mathbf{v} = \beta\mathbf{u} + \alpha\mathbf{v}.$$

Therefore,

$$A(\mathbf{u} - \mathbf{iv}) = \mathbf{A}(\mathbf{u} + \mathbf{i}(-\mathbf{v})) = \mathbf{Au} - \mathbf{iAv} = \alpha\mathbf{u} - \beta\mathbf{v} - (\beta\mathbf{u} + \alpha\mathbf{v}) = (\alpha - \beta\mathbf{i})(\mathbf{u} - \mathbf{v}).$$

Therefore, $\alpha - \beta i$ is an eigenvalue, and $\mathbf{u} - \mathbf{iv}$ is an associated eigenvector.

6. Suppose $\{\mathbf{b_1}, \mathbf{b_2}, \cdots, \mathbf{b_n}\}$ is a basis of V. Consider the set

$$\{T(\mathbf{b_1}), \mathbf{T}(\mathbf{b_2}), \cdots, \mathbf{T}(\mathbf{b_n})\}.$$

If

$$c_1 T(\mathbf{b_1}) + \mathbf{c_2}\mathbf{T}(\mathbf{b_2}) + \cdots + \mathbf{c_n}\mathbf{T}(\mathbf{b_n}) = \mathbf{0},$$

then because T is linear, we have

$$T(c_1\mathbf{b_1} + \mathbf{c_2}\mathbf{v_2} + \cdots + \mathbf{c_n}\mathbf{v_n}) = \mathbf{0}.$$

Since $T(\mathbf{0}) = \mathbf{0}$ and T is 1-1, we must have

$$c_1\mathbf{b_1} + \mathbf{c_2}\mathbf{v_2} + \cdots + \mathbf{c_n}\mathbf{v_n} = \mathbf{0}.$$

Since $\{\mathbf{b_1}, \mathbf{b_2}, \cdots, \mathbf{b_n}\}$ is a basis, it must be linearly independent, so $c_1 = c_2 = \cdots = c_n$.

Thus, $[\{T(\mathbf{b_1}), \mathbf{T}(\mathbf{b_2}), \cdots, \mathbf{T}(\mathbf{b_n})\}$ is linearly independent. Either it is a basis, or it is not. If it is a basis, then, the dimensions are the same. If it is not, then since it is linearly independent, we can add a vector to the set while maintaining its linear independence. Hence, the dimension of V is less than the dimension of W.

7. Notice that the eigenvalues are ± 1, and 1 has multiplicity 2. The eigenvectors for $\lambda = 1$ can be found by solving

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix}.$$

It is clear that there are two pivots and only one free variable. Thus, the dimension of the eigenspace is only one, and the matrix cannot be diagonalized.

8. From the Fundamental Subspaces theorem,

$$dim((\text{Nul}\, A)^{\perp}) = dim(\text{Row}(A)).$$

Also,

$$dim((\text{Nul}\, A^T) \perp) = dim(\text{Col}(A)).$$

Since the column space and the row space have the same dimension, we are done.

9. We need vectors $(a, b, c)^T$ whose dot product with those in the spanning set are 0. Thus,

$$a + b - 5c = 0$$

and

$$2a - c = 0.$$

This leads to the augmented matrix

$$\begin{bmatrix} 1 & 1 & -5 & 0 \\ 2 & 0 & -1 & 0 \end{bmatrix}$$

whose reduced echelon form is

$$\begin{bmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -9/2 & 0 \end{bmatrix}.$$

Thus, we see that $(-1, 9, 2)^T$ is a basis for $W^\perp$.

10. Since A has 6 free variables, it only has 14 pivots. Thus, there is a non-pivot row, and so we cannot guarantee a solution.

11. If $P^{-1}AP = B$, and $A\mathbf{x} = \lambda\mathbf{x}$ for a nonzero $\mathbf{x}$, then

$$APP^{-1}\mathbf{x} = \lambda\mathbf{x},$$

and

$$P^{-1}APP^{-1}\mathbf{x} = \mathbf{P^{-1}}\lambda\mathbf{x} = \lambda\mathbf{P^{-1}x},$$

or

$$B(P^{-1}\mathbf{x}) = \lambda(\mathbf{P^{-1}x}).$$

Thus, λ is an eigenvalue of B.

12. Notice that

$$W = span\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

and the spanning set is orthogonal. The projection of $\mathbf{y}$ onto W is

$$\frac{6}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{3}{1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}!$$

What is going on? $\mathbf{y} \in \mathbf{W}$, so it is its own projection.

13. a) The eigenvalues are 5, multiplicity 1, with eigenvector $(1, 2, 1)^T$, and 3, multiplicity 2, with eigenvectors $(1, 0, -1)^T$ and $(0, 1, 0)^T$. b) Omitted.