Name:______________________________

Student ID(see bubble sheet):________________

Section:______________________________

Instructor:

Math 313 (Linear Algebra)
Exam 3 Practice
Mar 26,27,28

Instructions:

- For questions which require a written answer, show all your work. Full credit will be given only if the necessary work is shown justifying your answer.
- Simplify your answers.
- Scientific calculators are allowed.
- Should you have need for more space than is allocated to answer a question, use the back of the page the problem is on and indicate this fact.
- Please do not talk about the test with other students until after the last day to take the exam.

Part I: Multiple Choice Questions: *Mark **all** answers which are correct for each question. (4 points each.)*

1. On $\mathbb{P}_2$ the linear transformation $T(p)(x) := x^2 p''(x)$ is given. Which of the following statements are true?

   a) T is invertible.

   b) T has a kernel of dimension 2.

   c) T is onto.

   d) The matrix of T w.r.t. the basis $B := \{1, t, t^2\}$ assumes the form
   $$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

   e) The matrix of T w.r.t. the basis $B := \{1, t, t^2\}$ assumes the form
   $$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   f) The matrix of T w.r.t. the basis $B := \{1, t, t^2\}$ cannot be a square matrix.

2. Let $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. Which of the following sets of vectors span the subspace $W^\perp$? Select all that apply:

   a) $\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \right\}$

   b) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

   c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

   d) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

   e) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

   f) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} \right\}$

3. Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$. Then, the matrix P that diagonalizes A is given by

   a) $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

   b) $\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$

   c) $\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}$

   d) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

   e) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

   f) $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

Part II: *Fill in the blank with the* ***best*** *possible answer. (4 points each.)*

4. (10 points)

   (a) A matrix A is called *symmetric* if $A^T = A$. Let M denote the vector space of all symmetric 2×2 matrices. Then $\dim M =$ _______________.

   (b) Fill in the matrix so that it has rank 1.

$$\begin{bmatrix} 1 & & 2 \\ 2 & -4 & \\ & -6 & \end{bmatrix}$$

   (c) Let $\{\mathbf{u_1}, \dots, \mathbf{u_p}\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\mathbf{y}$ in W, the weights of the linear combination
   $\mathbf{y} = c_1\mathbf{u_1} + \cdots + c_p\mathbf{u_p}$ are given by

$$c_j = \underline{\qquad\qquad} \qquad \text{for } j = 1, \dots, p.$$

   (d) Let $\mathbf{y} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$, and $W = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right\}$. The distance from $\mathbf{y}$ to W is ____________________.

   (e) If A is an $n \times n$ matrix, $\mathbf{x}$ is a nonzero vector, and $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ, then ____________________

   (f) Assume A is an $n \times n$ matrix that has distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_r$ with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_r$. What can be said of the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_r\}$? ____________________

Part III: *Justify your answer and show all work for full credit.*

5. For a real 2x2 matrix A prove or disprove by giving a counterexample that with every complex eigenvector $\mathbf{u} + \mathbf{iv}$ also its conjugate, $\mathbf{u} - \mathbf{iv}$, is an eigenvector of A.

6. Let $T : V \to W$ be a linear transformation that is one-to-one. Prove that $\dim V \leq \dim W$. *Hint: take a basis* $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ *of* V*, and consider the set* $\{T(\mathbf{b}_1), \ldots, T(\mathbf{b}_n)\}$ *in* W*.*

7. Can
$$F = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$
be diagonalized? If it can, find the diagonalizing matrix. Otherwise, show why it cannot.

8. Prove that for any $m \times n$ matrix A, we have $\dim(\operatorname{Nul} A)^{\perp} = \dim(\operatorname{Nul} A^T)^{\perp}$.

9. Let $W = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$. Find a basis for $W^{\perp}$.

10. Let A be a 15×20 matrix. For a particular vector $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ has 6 free variables. Can we guarantee that $A\mathbf{x} = \mathbf{b}$ always has a solution for any vector $\mathbf{b}$? Justify your conclusion.

11. Prove that if A and B are similar matrices, then A and B have the same eigenvalues.

12. Let $W = \left\{ \begin{bmatrix} a \\ b \\ a \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$. Use an orthogonal basis for W and the projection formula to find the closest point in W to $\mathbf{y} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$. Does your answer makes sense? Explain.

13. For the matrix A given by
$$A = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}$$
   (a) Find its eigenvalues and eigenvectors
   (b) Check that the eigenvalues and eigenvectors of A found in part (a) are correct by verifying that $A\mathbf{v} = \lambda\mathbf{v}$ for every eigenvalue λ and its corresponding eigenvector $\mathbf{v}$.

END OF EXAM