Math 313 Lecture #1
§1.1: Systems of Linear Equations

A **linear equation** in n variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where b and the **coefficients** $a_1, a_2, \ldots, a_n$ are numbers.

A **system of linear equations** (or simply **system**) is m linear equations in n variables:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m, \end{array}$$

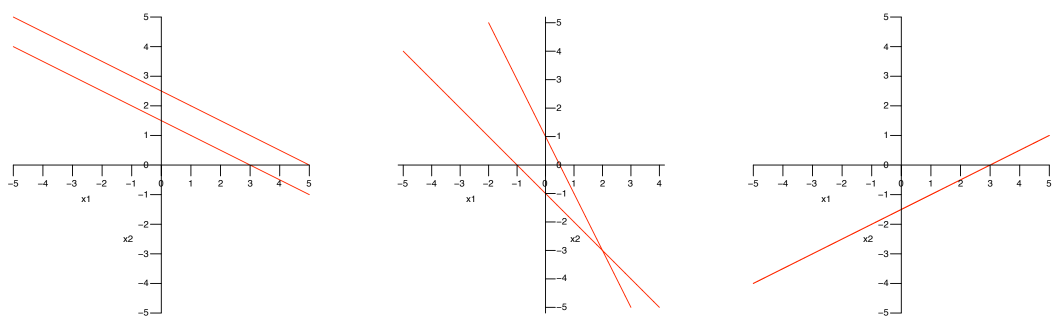
where the b_j and the a_{ij}'s are numbers.

A **solution** of a linear system is an ordered n-tuple of numbers $(s_1, s_2, \ldots, s_n)$ that satisfy all m equations simultaneously.

Example. Consider the three linear systems:

$$\begin{array}{ccccc} 2x_1 & + & 4x_2 & = & 6, \\ x_1 & + & 2x_2 & = & 5, \end{array} \qquad \begin{array}{ccccc} 2x_1 & + & x_2 & = & 1, \\ x_1 & + & x_2 & = & -1, \end{array} \qquad \begin{array}{ccccc} x_1 & - & 2x_2 & = & 3, \\ 2x_1 & - & 4x_2 & = & 6. \end{array}$$

Each linear equation here can be represented as a line in the plane:



The ordered pair $(2, -3)$ is a solution for the second system. [Check it ✓.] And there are no other solutions.

The ordered pair $(3, 0)$ is a solution for the third system. Actually, an ordered pair of the form $(s_1, s_2) = (3 + 2t, t)$ for any real t is a solution for the third system. [Check them ✓.]

However, the first systems has no solution. Why? Because if the ordered pair (s_1, s_2) is a solution, then $s_1 = 5 - 2s_2$ *and* $2s_1 = 6 - 4s_2$; but the second of these can be written as $s_1 = 3 - 2s_2$, so that $5 - 2s_2 = s_1 = 3 - 2s_2$ which implies that $5 = 3$. ////

The **solution set** of a linear system is the collection of all of its solutions.

The three systems in the above example illustrate what is possible for the solution set: no solutions (the empty set), exactly one solution, or infinitely many solutions.

A linear system is **consistent** if it has at least one solution; otherwise it is **inconsistent**.

To solve a consistent linear system is to find all of its solutions.

How is the solution set of a consistent linear system found?

Example. Consider the two linear systems:

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & 2x_3 & = & 1, \\ & & 2x_2 & & & = & 4, \\ & & & & x_3 & = & -1, \end{array} \qquad \begin{array}{rcrcrcr} x_1 & + & x_2 & + & 2x_3 & = & 1, \\ 2x_1 & + & 4x_2 & + & 4x_3 & = & 6, \\ -x_1 & - & x_2 & - & x_3 & = & -2. \end{array}$$

The first system is in **triangular form** and is easily solved: $x_3 = -1$, $x_2 = 2$, and $x_1 = 1 - x_2 - 2x_3 = 1 - 2 + 2 = 1$; so *the* (one and only) solution of this system is $(1, 2, -1)$.

A solution (x_1, x_2, x_3) of the second system is a solution of each of the equations in the second system; thus it is a solution of any new equation formed

1. by switching the order of the equations in the system,
2. by a nonzero multiple of any of the equations in the system, or
3. by adding a multiple of one of the equations in the system to another.

Adding the first and the third equations in the second system gives $x_3 = -1$.

Adding minus two times the first equation to the second gives $2x_2 = 4$, so that $x_2 = 2$.

The first equation in the second system then gives $x_1 = 1$.

The solution of the second system is $(1, 2, -1)$. [Check that $(1, 2, -1)$ is a solution of the second system ✓.]

The solution set of the second system is the same as the first system. ////

Two linear systems involving the *same* number of variables are said to be **equivalent** if they have the same solution set.

The above example illustrates that operations 1, 2, and 3 listed above can be applied to a system to obtain an equivalent system that is easier to solve.

When applying the operations 1, 2, and 3 to a linear system, the numbers a_{ij}'s and b_i's change, but the *placement* of the variables x_i's and the right hand side do not.

Only the numbers in the system need to be tracked while obtaining an equivalent system.

The numbers in a linear system can be encoded by the **augmented matrix**,

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{n1} & b_1 \\ a_{21} & a_{22} & \dots & a_{n2} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right].$$

The a_{ij} part of this is called the **coefficient matrix** and its **size** is $m \times n$ (m rows and n columns).

[The text does not have the vertical line segments in the augmented matrix separating the coefficients a_{ij} from the b_j.]

Operations 1, 2, and 3 for a linear system correspond to the **elementary row operations** on its augmented matrix:

1. interchange two rows,
2. multiply a row through by a nonzero real number,
3. replace a row by its sum with a multiple of another row.

With R_i representing the i^{th} row of the augmented matrix, the elementary row operations are symbolically 1. $R_i \leftrightarrow R_j$, 2. $\alpha R_i \to R_i$ for nonzero α, and 3. $R_i + \alpha R_j \to R_i$.

Applying these elementary row operations to the augmented matrix of a linear system may lead to an equivalent system that is easier to solve than the original.

Example. Find a linear system in triangular form that is equivalent to the linear system,

$$\begin{array}{rcrcrcl} & & 4x_2 & - & 3x_3 & = & 8, \\ 2x_1 & & & + & x_3 & = & 2, \\ 2x_1 & + & 2x_2 & & & = & 5. \end{array}$$

The augmented matrix for this linear system is

$$\begin{bmatrix} 0 & 4 & -3 & | & 8 \\ 2 & 0 & 1 & | & 2 \\ 2 & 2 & 0 & | & 5 \end{bmatrix}.$$

With the goal of a triangular form for the equivalent linear system, we apply row operations:

$$\begin{aligned} \begin{bmatrix} 0 & 4 & -3 & | & 8 \\ 2 & 0 & 1 & | & 2 \\ 2 & 2 & 0 & | & 5 \end{bmatrix} R_1 \leftrightarrow R_2 &\Leftrightarrow \begin{bmatrix} 2 & 0 & 1 & | & 2 \\ 0 & 4 & -3 & | & 8 \\ 2 & 2 & 0 & | & 5 \end{bmatrix} R_3 - R_1 \to R_3 \\ &\Leftrightarrow \begin{bmatrix} 2 & 0 & 1 & | & 2 \\ 0 & 4 & -3 & | & 8 \\ 0 & 2 & -1 & | & 3 \end{bmatrix} R_3 - (1/2)R_2 \to R_2 \\ &\Leftrightarrow \begin{bmatrix} 2 & 0 & 1 & | & 2 \\ 0 & 4 & -3 & | & 8 \\ 0 & 0 & 1/2 & | & -1 \end{bmatrix}. \end{aligned}$$

The the original linear system is equivalent to the linear system

$$\begin{array}{rcrcrcl} 2x_1 & & & + & x_3 & = & 2, \\ & & 4x_2 & - & 3x_3 & = & 8, \\ & & & & (1/2)x_3 & = & -1. \end{array}$$

Is the original linear system consistent? Yes, it is because $s_1 = 2$, $s_2 = 1/2$, $s_3 = -2$ is solution of it. ////

Two matrices (of the same size) are **row equivalent** if there a finite number of elementary row operations transforms one into the other.

If augmented matrices for two linear systems are row equivalent, then the two linear systems have the same solutions set (and hence are equivalent).

There are two fundamental questions to be answered about a linear system: (1) when is it consistent? and (2) if a solution exists, is it unique?

Through row operations, we will completely answer these two fundamental questions in the next lecture.