Math 313 Lecture #2
§1.2: Row Reduction and Echelon Forms

Row reduction is the use of a finite number of elementary row operations to transform the augmented matrix of a linear system (or any rectangular matrix) into a triangular form.

The questions of existence and uniqueness of solutions, and what the solution set is when solutions exist, are all answered by the triangular form, so we pay particular attention to it.

A row of a rectangular matrix is a **nonzero row** when at least one entry of the row is nonzero, and the **leading entry** of a nonzero row is the left-most nonzero entry of the row.

A rectangular matrix is said to be in **echelon form** if

1. all nonzero rows are above any rows of all zeros,
2. each leading of a nonzero row is in a column to the right of the leading entry of the row above it, and
3. all entries in a column below a leading entry are zero.

A rectangular matrix is said to be in **reduced echelon form** (or reduced row echelon form) if it is echelon form and

4. the leading entry of each nonzero row is 1, and
5. each leading entry is the only nonzero entry in its column.

Example. Which of the following matrices are in row echelon form? reduced echelon form? Neither?

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 6 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

If neither, what row operation is needed to bring it into echelon form? / / / /

Any nonzero rectangular matrix A (i.e., at least one entry is nonzero) can be row reduced to more than one echelon form U (just multiply a nonzero row through by a nonzero scalar).

It might not be surprising that there is only one reduced echelon form obtained by row reduction of a matrix.

Theorem 1. Each nonzero matrix is row equivalent to exactly one reduced echelon matrix.

When row reducing an echelon form to reduced echelon form, the leading entries of the nonzero rows are always in the same positions.

A **pivot position** in a matrix A is a location (in terms row and column) in A that corresponds to a leading 1 in the reduced echelon form U of A, and a **pivot column** of A is a column of A that contains a pivot position.

We already being using pivot positions in row reducing a matrix into an echelon form.

Example. Find an echelon form for the augmented matrix for

$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & + & 2x_3 & - & x_4 & = & 1, \\ -2x_1 & - & 2x_2 & - & 3x_3 & + & 5x_4 & = & 0, \\ x_1 & + & x_2 & + & 3x_3 & + & 2x_4 & = & 3, \\ & & & & x_3 & + & 3x_4 & = & 6. \end{array}$$

We apply the row reduction to the augmented matrix for this system:

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 1 \\ -2 & -2 & -3 & 5 & 0 \\ 1 & 1 & 3 & 2 & 3 \\ 0 & 0 & 1 & 3 & 6 \end{array}\right] \begin{array}{l} \\ R_2 + 2R_1 \to R_2 \\ R_3 - R_1 \to R_3 \\ \\ \end{array}$$

$$\Leftrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 3 & 6 \end{array}\right] \begin{array}{l} \\ \text{no nonzero pivot position in second column below first row} \\ R_3 - R_2 \to R_3 \\ R_4 - R_2 \to R_4 \end{array}$$

$$\Leftrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{array}\right] \begin{array}{l} \\ \\ \\ (1/4)R_4 \to R_4 \text{ followed by } R_3 \leftrightarrow R_4 \end{array}$$

$$\Leftrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Reinserting the variables in the last row of this augmented matrix gives

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 0$$

which is *always* true.

Reinserting the variables in the third row gives

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 1$$

which is *never* true.

Thus the system corresponding to the last augmented matrix is inconsistent.

Since the elementary row operations preserve the solution set (they give equivalent systems), the original system is inconsistent as well. ////

We now show how to parametrically describe the solution set of a consistent linear system when there is more than one solution.

Example. Solve

$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & + & 2x_3 & - & x_4 & = & 1, \\ -2x_1 & - & 2x_2 & - & 3x_3 & + & 5x_4 & = & 0, \\ x_1 & + & x_2 & + & 3x_3 & + & 2x_4 & = & 3, \\ & & & & x_3 & + & 3x_4 & = & 2. \end{array}$$

The only difference between this and the previous example is the fourth equation: $x_3 + 3x_4 = 2$ instead of $x_3 + 3x_4 = 6$.

By row reduction applied to the augmented matrix of the system we get an echelon form

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Reinserting the variables into the first and second rows gives

$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & + & 2x_3 & - & x_4 & = & 1, \\ & & & & x_3 & + & 3x_4 & = & 2. \end{array}$$

The first variable in each row corresponds to a pivot column of the augmented matrix, and is called a **basic** variable.

The remaining variables are called **free** variables.

Moving the free variables to the right hand side gives

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_3 & = & 1 & - & x_2 & + & x_4, \\ & & x_3 & = & 2 & - & 3x_4. & & \end{array}$$

The left hand side of this is in triangular form, so that for each pair of values assigned to x_2 and x_4, there is a unique solution of the system.

But since x_2 and x_4 are free (they can be assigned *any* value), the systems has infinitely many solutions.

Assigning $x_2 = t$ and $x_4 = s$ gives by **back substitution**

$$x_3 = 2 - 3s,$$
$$x_1 = 1 - t + s - 2x_3 = 1 - t + s - 2(2 - 3s) = -3 - t + 7s.$$

Varying the values of t and s (the *free* variables) gives all of the solutions of the system, and hence the solution set is

$$\{(-3 - t + 7s, t, 2 - 3s, s) : t, s \in \mathbb{R}\}.$$

The original system is therefore consistent, having infinitely many solutions. ////

We can now answer the two fundamental questions (1) when is a linear system consistent? and (2) if a solution exists, is it unique?

Theorem 2. (Existence and Uniqueness) A linear system is consistent if and only if the rightmost column of its augmented matrix is not a pivot column, i.e., an echelon form of the augmented matrix doe not contain a row of the form

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & | & b \end{bmatrix}$$

with a nonzero b. If the linear system is consistent, then it has a unique solution when there are no free variables, and infinitely many solutions when there is at least one free variable.