Math 313 Lecture #3
§1.3: Vector Equations

Vectors. A matrix with only one column and n rows is called a **column vector**, or simply a **vector**, and is an ordered n-tuple:

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \ \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}, \text{ etc.}$$

[In these notes and on the whiteboard, the notation $\vec{u}$ will be used instead of $\mathbf{u}$ for vector.]

We say that such a vector $\vec{u}$ belongs to the set $\mathbb{R}^n$.

Two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are **equal** when their corresponding entries are equal, i.e., $u_i = v_i$ for all $i = 1, \ldots, n$.

The **sum** of two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ is the vector

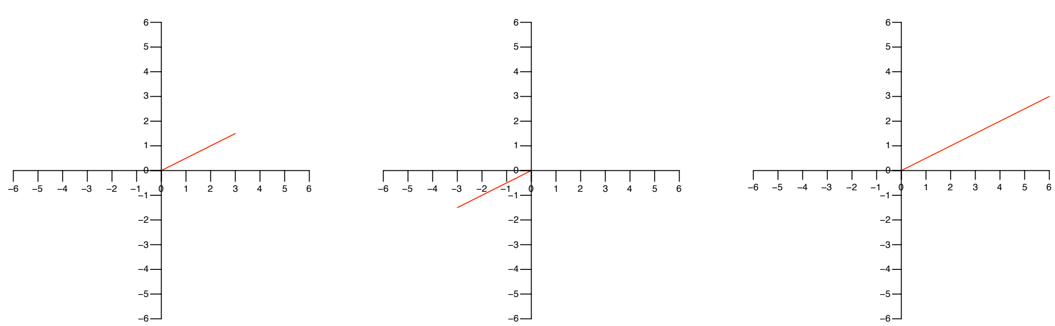
$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}.$$

The **scalar multiple** of a vector $\vec{u}$ by a real scalar c is the vector

$$c\vec{u} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}.$$

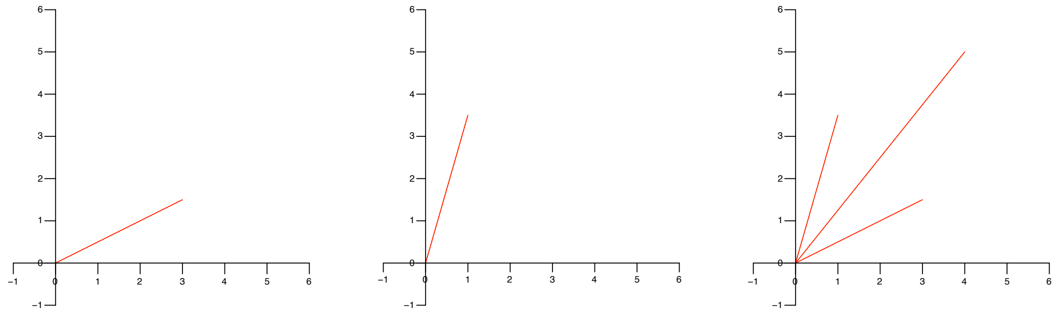
Example. We can identify a vector in $\mathbb{R}^2$ with a geometric point. Here is a vector along with two scalar multiples.

$$\vec{u} = \begin{bmatrix} 3 \\ 1.5 \end{bmatrix}, \qquad -\vec{u} = \begin{bmatrix} -3 \\ -1.5 \end{bmatrix}, \qquad 2\vec{u} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}.$$



Example. Here are two vectors in $\mathbb{R}^2$ along with their sum.

$$\vec{u} = \begin{bmatrix} 3 \\ 1.5 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} 1 \\ 3.5 \end{bmatrix}, \qquad \vec{u} + \vec{v} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$



The sum of two vectors follows the parallelogram rule for addition: the sum is the fourth vertex of the parallelogram with vertices $\vec{0}$ (the zero vector all of whose entries are zero), $\vec{u}$, and $\vec{v}$.

Algebraic Properties of Vectors. Vectors in $\mathbb{R}^n$ enjoy several algebraic properties reminiscent of those of the real numbers.
For $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $\mathbb{R}^n$ and c and d in $\mathbb{R}$, these are

(i) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$,

(ii) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$,

(iii) $\vec{u} + \vec{0} = \vec{u}$,

(iv) $\vec{u} + (-\vec{u}) = \vec{0}$, where $-\vec{u}$ is the scalar multiple $(-1)\vec{u}$,

(v) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$,

(vi) $(c + d)\vec{u} = c\vec{u} + d\vec{u}$,

(vii) $(cd)\vec{u} = c(d\vec{u})$, and

(viii) $1\vec{u} = \vec{u}$.

We almost always write $\vec{u} + (-1)\vec{v}$ as $\vec{u} - \vec{v}$. What does $\vec{u} - \vec{v}$ look like?

Linear Combinations. The **linear combination** of p vectors $\vec{v}_1, \vec{v}_2, \ldots, v_p$ in $\mathbb{R}^n$ with **weights** $c_1, c_2, \ldots, c_p$ is the vector

$$\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p.$$

There is no need for parenthesis in a linear combination because of Property (iii).

In the previous Example, what do the linear combinations $2\vec{u} + \vec{v}$ and $\vec{u} - 2\vec{v}$ look like?

Example. For vectors in $\mathbb{R}^3$,

$$\vec{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \ \vec{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix},$$

do there exist weights x_1 and x_2 so that $\vec{b}$ is the linear combination

$$x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}?$$

The entries of the linear combination make a linear system

$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & 7, \\ -2x_1 & + & 5x_2 & = & 4, \\ -5x_1 & + & 6x_2 & = & -3. \end{array}.$$

Row reducing the augment matrix for this linear systems gives

$$\begin{bmatrix} 1 & 2 & | & 7 \\ -2 & 5 & | & 4 \\ -5 & 6 & | & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & | & 7 \\ 0 & 9 & | & 18 \\ 0 & 16 & | & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & | & 7 \\ 0 & 1 & | & 2 \\ 0 & 16 & | & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & | & 3 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & 0 \end{bmatrix}.$$

Thus $x_1 = 3$ and $x_2 = 2$ satisfies $x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}$, as can be verified. ////

The original augmented matrix has the form of $\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & | & \vec{b} \end{bmatrix}$.

The **vector equation** $x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}$ as the same solution set as that of the linear system whose augmented matrix is $\begin{bmatrix} a_1 & a_2 & | & \vec{b} \end{bmatrix}$.

The existence of a solution x_1 and x_2 is the same as saying that $\vec{b}$ *is* a linear combination of the vectors $\vec{a}_1$ and $\vec{a}_2$.

This brings us to an important concept in linear algebra.

Definition. For p vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^n$, the set of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ is called the **subset of** $\mathbb{R}^n$ **spanned** (or generated) **by** $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$, and we write $\text{Span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p)$ for this subset.

Examples. (a) For a nonzero vector $\vec{v}$ in $\mathbb{R}^2$, what is $\text{Span}(\vec{v})$?

The linear combination of one vector is just the set of all scalar multiples $c\vec{v}$, and so $\text{Span}(\vec{v})$ is a line through the origin.

(b) For vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$ with $\vec{v}$ not a scalar multiple of $\vec{u}$, we have that $\text{Span}(\vec{u}, \vec{v})$ is $\mathbb{R}^2$ (think of an "etch-a-sketch").

(c) For vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ with $\vec{v}$ not a scalar multiple of $\vec{u}$, we have that $\text{Span}(\vec{u}, \vec{v})$ is a plane through the origin.

And the question of if there are scalars x_1 and x_2 or not such that $x_1\vec{u} + x_2\vec{v} = \vec{b}$ is about if $\vec{b}$ is in $\text{Span}(\vec{u}, \vec{v})$ or not.

Example. Is the vector

$$\vec{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$$

in subset of $\mathbb{R}^3$ spanned by the vectors

$$\vec{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \ \vec{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}?$$

Since $\vec{a}_2$ is not a scalar multiple of $\vec{a}_1$, then $\text{Span}(\vec{a}_1, \vec{a}_2)$ is a plane through the origin in $\mathbb{R}^3$.

The question of whether or not $\vec{b}$ is in this plane can be answered by row reducing the augmented matrix associated with the linear combination $x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}$:

$$\begin{bmatrix} 1 & 5 & | & -3 \\ -2 & -13 & | & 8 \\ 3 & -3 & | & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & | & -3 \\ 0 & -3 & | & 2 \\ 0 & -18 & | & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & | & -3 \\ 0 & -3 & | & 2 \\ 0 & 0 & | & -2 \end{bmatrix}.$$

Because of the appearance of the row $\begin{bmatrix} 0 & 0 & | & -2 \end{bmatrix}$, the linear system is inconsistent, and so $\vec{b}$ is not in $\text{Span}(\vec{a}_1, \vec{a}_2)$.