Math 313 Lecture #5
§1.5: Solution Sets of Linear Systems

We have seen how to parametrically describe a solution set of a consistent linear system.

We will advance this parametric description through the use of linear combinations.

We start with a linear system that is always consistent.

Homogeneous Systems. A linear system is said to be **homogeneous** if it can be written in the form $A\vec{x} = \vec{0}$, where A is an $m \times n$ matrix and $\vec{0}$ is the zero vector in $\mathbb{R}^n$.

A homogeneous system always has the **trivial solution** $\vec{x} = \vec{0}$ in $\mathbb{R}^n$ because $A\vec{0} = \vec{0}$.

[We are using the same notation $\vec{0}$ to represent the zero vector in $\mathbb{R}^m$ and in $\mathbb{R}^n$. Typically the context determines the number of entries in $\vec{0}$.]

An important question then for a homogeneous system is whether the trivial solution is the only solution (is it unique) or there are other **nontrivial solutions**.

The answer to this question follows from the Uniqueness Theorem: $A\vec{x} = \vec{0}$ has a nontrivial solution if and only if there is a free variable.

Example. Find the solution set of $A\vec{x} = \vec{0}$ where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}.$$

We row reduce the augmented matrix to reduced echelon form:

$$\begin{bmatrix} 3 & 5 & -4 & | & 0 \\ -3 & -2 & 4 & | & 0 \\ 6 & 1 & -8 & | & 0 \end{bmatrix} \begin{matrix} \\ R_2 + R_1 \to R_2 \\ R_3 - 2R_1 \to R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 3 & 5 & -4 & | & 0 \\ 0 & 3 & 0 & | & 0 \\ 0 & -9 & 0 & | & 0 \end{bmatrix} \begin{matrix} \\ \\ R_3 + 3R_2 \to R_3 \end{matrix}$$

$$= \begin{bmatrix} 3 & 5 & -4 & | & 0 \\ 0 & 3 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \begin{matrix} \\ (1/3)R_2 \to R_2 \\ \\ \end{matrix}$$

$$= \begin{bmatrix} 3 & 5 & -4 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \begin{matrix} R_1 - 5R_2 \to R_1 \\ \\ \\ \end{matrix}$$

$$= \begin{bmatrix} 3 & 0 & -4 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \begin{matrix} (1/3)R_1 \to R_1 \\ \\ \\ \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & -4/3 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

Now, did we really need that last column in the augmented matrix as we row reduced? No, because it stays the same under all row operations.

The variables x_1 and x_2 are basic and the variable x_3 is free.

Setting $x_3 = t$, we obtain $x_2 = 0$ and $x_1 = 4t/3$.

The solution set is parametrically given by $\{(4t/3, 0, t) : t \in \mathbb{R}\}$.

To realize this solution set as linear combinations, we write each solution in vector form and use algebraic properties of vectors:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4t/3 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}.$$

Geometrically, the solution set is a line through the origin $\vec{0}$ in $\mathbb{R}^3$, with the trivial solution corresponding to $t = 0$.

Example. Find the solution set of $A\vec{x} = \vec{0}$ where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}.$$

Since we are finding the solution set of a homogeneous system, we need only row reduce the matrix A:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \to R_2 \\ R_3 - 3R_1 \to R_3 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We have one basic variable x_1 and two free variables $x_2 = s$ and $x_3 = t$.

The basic variable in terms of the free variables is

$$x_1 = -2x_2 - 3x_3 = -2s - 3t.$$

So solutions of the homogeneous system as linear combinations are

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2s - 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

Since the two vectors here are not scalar multiples of each other, the solution set is a plane through the origin in $\mathbb{R}^3$. ////

In both of the examples, we saw that the solution set of a homogeneous system $A\vec{x} = \vec{0}$ can be written in the **parametric vector form**,

$$\vec{x} = t_1\vec{u}_1 + t_2\vec{u}_2 + \cdots t_p\vec{u}_p.$$

where each $\vec{u}_j$ satisfies $A\vec{u}_j = \vec{0}$. (You should check this in the examples.)

Solutions Sets of Consistent Nonhomogeneous Systems. For a nonhomogeneous system $A\vec{x} = \vec{b}$ (where $\vec{b} \neq \vec{0}$), we do not have that $\vec{0}$ is a solution because $A\vec{0} = \vec{0}$ is not $\vec{b}$.

But the solution set of a consistent $A\vec{x} = \vec{b}$ is related to the solution set of $A\vec{x} = \vec{0}$.

Example We will go back to the 3×3 matrix of the first example,

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix},$$

and find the solution set of $A\vec{x} = \vec{b}$ where

$$\vec{b} = \begin{bmatrix} -6 \\ 3 \\ -3 \end{bmatrix}.$$

We row reduce the augmented matrix:

$$\begin{aligned}
&\begin{bmatrix} 3 & 5 & -4 & | & -6 \\ -3 & -2 & 4 & | & 3 \\ 6 & 1 & -8 & | & -3 \end{bmatrix} \begin{matrix} \\ R_2 + R_1 \to R_2 \\ R_3 - 2R_1 \to R_3 \end{matrix} \\
\sim &\begin{bmatrix} 3 & 5 & -4 & | & -6 \\ 0 & 3 & 0 & | & -3 \\ 0 & -9 & 0 & | & 9 \end{bmatrix} \begin{matrix} \\ \\ R_3 + 3R_2 \to R_3 \end{matrix} \\
= &\begin{bmatrix} 3 & 5 & -4 & | & -6 \\ 0 & 3 & 0 & | & -3 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \begin{matrix} \\ (1/3)R_2 \to R_2 \\ \\ \end{matrix} \\
= &\begin{bmatrix} 3 & 5 & -4 & | & -6 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \begin{matrix} R_1 - 5R_2 \to R_1 \\ \\ \\ \end{matrix} \\
= &\begin{bmatrix} 3 & 0 & -4 & | & -1 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \begin{matrix} (1/3)R_1 \to R_1 \\ \\ \\ \end{matrix} \\
= &\begin{bmatrix} 1 & 0 & -4/3 & | & -1/3 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}.
\end{aligned}$$

With x_1 and x_2 as basic variables and $x_3 = t$ a free variable, the solutions are

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/3 + 4t/3 \\ -1 \\ t \end{bmatrix} = \begin{bmatrix} -1/3 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} = \vec{p} + t\vec{u}.$$

This is the **parametric vector equation** of a line that is parallel to the line $\vec{x} = t\vec{u}$.

Notice that $A\vec{p} = \vec{b}$ and $A\vec{u} = \vec{0}$.

Theorem 6. If $A\vec{x} = \vec{b}$ is consistent with $\vec{p}$ satisfying $A\vec{p} = \vec{b}$, then every solution of $A\vec{x} = \vec{b}$ is of the form $\vec{x} = \vec{p} + \vec{v}_h$ where $\vec{v}_h$ is any vector that satisfies $A\vec{v}_h = \vec{0}$.