Math 313 Lecture #6
§1.7: Linear Independence

We are going to the use the homogeneous system $A\vec{x} = \vec{0}$ to analyze a critically important concept for vectors in $\mathbb{R}^n$.

Consider in $\mathbb{R}^3$ the three vectors

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Notice that $\vec{v}_3 = \vec{v}_1 - \vec{v}_2$, and so one vector is a linear combination of the others.

Rewritten we have the linear combination $\vec{v}_1 - \vec{v}_2 - \vec{v}_3 = \vec{0}$, or as a matrix equation,

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Definitions. Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^n$ are said to be **linearly dependent** if there are weights $c_1, c_2, \ldots, c_p$, not all zero, such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0}.$$

This equation is called a **linear dependence relation** for linearly dependent vectors.

Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^n$ are said to be **linearly independent** if the *only* solution of

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = \vec{0}$$

is the trivial one, i.e., $x_1 = 0, \ x_2 = 0, \ldots, \ x_p = 0$.

Since we can switch between vector equations and matrix equations, the linear dependence or linear independence of vectors is connected with a homogeneous system.

For $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_p \end{bmatrix}$ we have that $A\vec{x} = \vec{0}$ has the same solution set as that of

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_p\vec{a}_p = \vec{0}.$$

So the columns of A are linearly independent if and only if the homogeneous system $A\vec{x} = \vec{0}$ has only the trivial solution.

Example Are the columns of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 4 \\ 6 & -2 & 23 \end{bmatrix}$$

linearly independent?

We answer the question by row reducing the matrix:

$$\begin{aligned}
&\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 4 \\ 6 & -2 & 23 \end{bmatrix} R_1 \leftrightarrow R_2 \\
\sim &\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 6 & -2 & 23 \end{bmatrix} R_3 - 6R_1 \to R_3 \\
\sim &\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & -2 & -1 \end{bmatrix} R_3 + 2R_2 \to R_3 \\
\sim &\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.
\end{aligned}$$

This says that there are no free variables, and so $A\vec{x} = \vec{0}$ has only the trivial solution, meaning that the columns of A are linearly independent. ////

A single vector $\vec{v}_1$ in $\mathbb{R}^n$ is linearly independent if and only if $\vec{v}_1 \neq \vec{0}$.

Why? Because $x_1\vec{v}_1 = \vec{0}$ implies that $x_1 = 0$ when $\vec{v} \neq \vec{0}$.

Two vectors $\vec{v}_1$ and $\vec{v}_2$ in $\mathbb{R}^n$ are linearly independent if and only if neither is a scalar multiple of the other.

Why? Because if $\vec{v}_1$ and $\vec{v}_2$ are linearly dependent, then $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$, without loss of generality (or WLOG for short) say $c_1 \neq 0$ gives

$$\vec{v}_1 = -\frac{c_2}{c_1}\vec{v}_2,$$

and so one vector is a scalar multiple of the other.

On the other hand, if one of $\vec{v}_1$ and $\vec{v}_2$ is a scalar multiple of the other, say WLOG $\vec{v}_2 = d\vec{v}_1$, then

$$-d\vec{v}_1 + \vec{v}_2 = \vec{0}$$

where the weight of $\vec{v}_2$ is not zero, and so $\vec{v}_1$ and $\vec{v}_2$ are linear dependent.

For three or more vectors, we can similarly understand when they are linearly dependent.

Theorem 7. Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^n$ are linearly dependent if and only if one of the p vectors is a linear combination of the other $p-1$ vectors.

Proof. WLOG, suppose that $\vec{v}_p$ is a linear combination of the other $p-1$ vectors: there are weights $\alpha_1, \alpha_2, \ldots, \alpha_{p-1}$ such that

$$\vec{v}_p = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_{p-1}\vec{v}_{p-1}.$$

Moving $\vec{v}_p$ to the other side gives

$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_{p-1}\vec{v}_{p-1} - \vec{v}_p = \vec{0}.$$

Letting $c_i = \alpha_i$ for $i = 1, \ldots, n-1$, and $c_p = -1$ gives

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_p = \vec{0}$$

where not all of the c_i's are zero, and thus the vectors $\vec{v}_1,\ \vec{v}_2, \ldots,\ \vec{v}_p$ are linearly dependent.

On the other hand, if the vectors $\vec{v}_1,\ \vec{v}_2, \ldots,\ \vec{v}_p$ are linearly dependent, then at least one of the c_i's in

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0},$$

can be chosen to be nonzero.

If WLOG we say it is c_p, then solving for $\vec{v}_p$ gives

$$\vec{v}_p = -\frac{c_1}{c_p}\vec{v}_1 - \frac{c_2}{c_p}\vec{v}_2 - \cdots - \frac{c_{p-1}}{c_p}\vec{v}_{p-1}.$$

Thus $\vec{v}_p$ is a linear combination of the other $p-1$ vectors. □

The contrapositive of Theorem 7 gives a characterization of linear independence: vectors $\vec{v}_1,\ \vec{v}_2, \ldots,\ \vec{v}_p$ in $\mathbb{R}^n$ are linearly independent if and only if none of the p vectors is a linear combination of the other $p-1$ vectors.

We explore some of the connections between the subset spanned by vectors and linear independence of vectors.

Example. Suppose $\vec{u}$ and $\vec{v}$ are linearly independent vectors in $\mathbb{R}^3$.

Could either $\vec{u}$ or $\vec{v}$ be the zero vector? No, because if say $\vec{u} = \vec{0}$, then the vectors $\vec{0}$ and $\vec{v}$ would be linearly dependent because $1\vec{0} + 0\vec{v} = \vec{0}$.

A similar argument holds if $\vec{v} = \vec{0}$.

We know that $\text{Span}(\vec{u}, \vec{v})$ is a plane through the origin in $\mathbb{R}^3$.

For a vector $\vec{w}$ in $\mathbb{R}^3$, what can we say about the linear dependence or linearly independence of the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$?

If $\vec{w}$ belongs to $\text{Span}(\vec{u}, \vec{v})$, then $\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$, and so by Theorem 7, the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly dependent.

On the other hand, if the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly dependent, then by Theorem 7, one of the three vectors is a linear combination of the other two.

There are two cases to consider.

Case 1. If $\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$, then $\vec{w}$ is in $\text{Span}(\vec{u}, \vec{v})$.

Case 2. WLOG, suppose that the nonzero $\vec{u}$ is a linear combination of $\vec{v}$ and $\vec{w}$.

Then there are weights c_1 and c_2 such that $\vec{u} = c_1\vec{v} + c_2\vec{w}$.

Could $c_2 = 0$? No because then $\vec{u}$ would be a scalar multiple of $\vec{v}$, which is a contradiction.

So we get $\vec{w} = (1/c_2)\vec{u} - (c_1/c_2)\vec{v}$, hence $\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$, meaning that $\vec{w}$ is in $\text{Span}(\vec{u}, \vec{v})$. / / / /

There are some situations where we can guarantee the linear dependence of vectors.

Theorem 8. For p vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^n$, if $p > n$, then the p vectors are linearly dependent.

Proof. In row reducing the $n \times p$ matrix $A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_p \end{bmatrix}$ there are more columns than pivot positions.

Thus there are free variables and hence nontrivial solutions of $A\vec{x} = \vec{0}$, meaning the columns of A are linearly dependent. □

Theorem 9. If one of the p vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{R}^n$ is the zero vector, then the set of p vectors is linearly dependent.

Proof. In the linear combination $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p$ with WLOG say $v_p = \vec{0}$, we can take the first $p - 1$ weights all to be zero while taking $c_p = 1$, thus making the linear combination sum to $\vec{0}$. □