Math 313 Lecture #7
§1.8: Introduction to Linear Transformations

The notation $A\vec{x} = \vec{b}$ for a system of linear equations has another more function-like interpretation.

We can think of $\vec{x}$ as an input and $A\vec{x}$ as an output, i.e., we have a function from $\vec{x}$ in $\mathbb{R}^n$ to $A\vec{x}$ in $\mathbb{R}^m$.

This expands the notion of function $y = f(x)$ from that of a real number x as an input and a real number y as an output, to a vector $\vec{x}$ as an input and a vector $\vec{y}$ as an output.

Definitions. A **transformation** (or function or mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each input $\vec{x}$ in $\mathbb{R}^n$ an output vector $T(\vec{x})$ in $\mathbb{R}^m$.

The set of inputs $\mathbb{R}^n$ is called the **domain** of T.

The set $\mathbb{R}^m$ where the outputs of T live, is called the **codomain** of T.

The notation $T : \mathbb{R}^n \to \mathbb{R}^m$ reminds of the domain and codomain of T.

For each $\vec{x}$ in the domain, the output $T(\vec{x})$ is called the **image** of $\vec{x}$ under T.

The set of all images,

$$\{T(\vec{x}) : \vec{x} \in \mathbb{R}^n\}$$

is a subset of the codomain of T, and is called the **range** of T.

Matrix Transformations. We focus mainly on those transformations of the form $T(\vec{x}) = A\vec{x}$. [Math 314 deals with those transformations not of this form.]

For an $m \times n$ matrix A, the domain of T is $\mathbb{R}^n$ and the codomain of T is $\mathbb{R}^m$.

We also write $\vec{x} \to A\vec{x}$ for the matrix transformation T.

Recalling that $A\vec{x}$ is the same as a linear combination of the columns of A, we infer that the range of T is the set of all linear combinations of the columns of A.

In other words, the range of $\vec{x} \to A\vec{x}$ is the subset of $\mathbb{R}^m$ spanned by the columns of A.

Basic questions about matrix transformations are: (1) for a given vector $\vec{b}$ in the codomain, is there a vector $\vec{x}$ in the domain with $A\vec{x} = \vec{b}$? (2) Is there more than one choice of $\vec{x}$ such that $A\vec{x} = \vec{b}$?

Example. Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 3 & -1 \end{bmatrix},\ \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},\ \vec{c} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}.$$

With the size of A being 3×2, the matrix transformation $\vec{x} \to A\vec{x}$ has domain $\mathbb{R}^2$ and codomain $\mathbb{R}^3$.

(a) Is there a vector $\vec{x}$ in $\mathbb{R}^2$ such that $A\vec{x} = \vec{b}$?

This can be answered by row reduction of the augmented matrix:

$$\begin{bmatrix} 2 & 1 & | & 1 \\ 1 & 0 & | & 2 \\ 3 & -1 & | & 3 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & | & 2 \\ 2 & 1 & | & 1 \\ 3 & -1 & | & 3 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \to R_2 \\ R_3 - 3R_1 \to R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & -3 \\ 0 & -1 & | & -3 \end{bmatrix} \begin{matrix} \\ \\ R_3 + R_2 \to R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & -3 \\ 0 & 0 & | & -6 \end{bmatrix}.$$

The last row tells us that the linear system $A\vec{x} = \vec{b}$ is inconsistent, so there is no $\vec{x}$ in $\mathbb{R}^2$ for which $A\vec{x} = \vec{b}$.

In terms of the new terminology about transformations we learned earlier, we know now that $\vec{b}$ is not in the range of the matrix transformation.

(b) Is there $\vec{x}$ in $\mathbb{R}^2$ such that $A\vec{x} = \vec{c}$?

We answer this by row reduction of the augmented matrix:

$$\begin{bmatrix} 2 & 1 & | & 4 \\ 1 & 0 & | & 1 \\ 3 & -1 & | & 1 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & | & 1 \\ 2 & 1 & | & 4 \\ 3 & -1 & | & 1 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \to R_2 \\ R_3 - 3R_1 \to R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 2 \\ 0 & -1 & | & -2 \end{bmatrix} \begin{matrix} \\ \\ R_3 + R_2 \to R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & 0 \end{bmatrix}.$$

So $\vec{c}$ is in the range of the matrix transformation.

Is there more than one choice of $\vec{x}$ for which $A\vec{x} = \vec{c}$? No, only one choice because there is no free variable.

Linear Transformations. A matrix transformation $\vec{x} \to A\vec{x}$ has the properties

$$A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} \text{ and } A(c\vec{u}) = c(A\vec{u}).$$

We name these properties.

Definition. A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called **linear** if (i) $T(\vec{u}+\vec{v}) = T(\vec{u})+T(\vec{v})$ for all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$, and (2) $T(c\vec{u}) = cT(\vec{u})$ for all scalars c and all $\vec{u}$ in $\mathbb{R}^n$.

Every matrix transformation is linear.

Linearity of a transformation implies that it has several other important properties.

If T is linear, then $T(\vec{0}) = \vec{0}$ because

$$T(\vec{0}) = T(0\vec{0}) = 0T(\vec{0}) = \vec{0}.$$

If T is linear then T satisfies the *superposition principle*, i.e.,

$$T(c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_p\vec{u}_p) = c_1T(\vec{u}_1) + c_2T(u_2) + \cdots + c_pT(u_p)$$

by induction because we have the base case

$$T(c_1\vec{u}_1 + c_2\vec{u}_2) = T(c\vec{u}_1) + T(c_2\vec{u}_2) = c_1T(\vec{u}_1) + c_2T(\vec{u}_2)$$

and the induction step

$$\begin{aligned} &T(c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_k\vec{u}_k + c_{k+1}\vec{u}_{k+1}) \\ &= T(c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_k\vec{u}_k) + T(c_{k+1}\vec{u}_{k+1}) \\ &= c_1T(\vec{u}_1) + c_2T(\vec{u}_2) + \cdots + c_kT(\vec{u}_k) + c_{k+1}T(\vec{u}_{k+1}). \end{aligned}$$

Geometry of Matrix transformations. We can visualize a matrix transformation especially when the codomain is the same as the domain.

Example. For $r > 0$ let

$$A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}.$$

Then the image of the matrix transformation $\vec{x} \to A\vec{x}$ is

$$A\vec{x} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix} = r \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = r\vec{x}.$$

This linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a **contraction** when $0 < r < 1$ and is a **dilation** when $r > 1$.

Example. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The image of the matrix transformation $\vec{x} \to A\vec{x}$ is

$$A\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$

This linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a **projection** onto the x_1-axis.

Example. Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

The image of the matrix transformation $\vec{x} \to A\vec{x}$ is

$$A\vec{x} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix}.$$

This linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a **shear**.

Example. Let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The image of the matrix transformation $\vec{x} \to A\vec{x}$ is

$$A\vec{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}.$$

This linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a **rotation**.