Math 313 Lecture #8
§1.9: The Matrix of a Linear Transformations

We have seen that every matrix transformation is a linear transformation.

We will show that the converse is true: every linear transformation is a matrix transformation; and we will show to find the matrix.

To do this we will need the columns of the $n \times n$ **identity matrix**

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

With $\vec{e}_j$ as the j^{th} column of I_n, we have for every $\vec{x}$ in $\mathbb{R}^n$ that

$$\vec{x} = I_n\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n.$$

Theorem 10. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then there is a unique $m \times n$ matrix A such that

$$T(\vec{x}) = A\vec{x}, \quad \vec{x} \in \mathbb{R}^n,$$

where the j^{th} column of A is vector $T(\vec{e}_j)$.

Proof. For $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$, it follows by the linearity of T that

$$\begin{aligned} T(\vec{x}) &= T(x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n) \\ &= x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + \cdots + x_nT(\vec{e}_n) \\ &= \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &= A\vec{x}. \end{aligned}$$

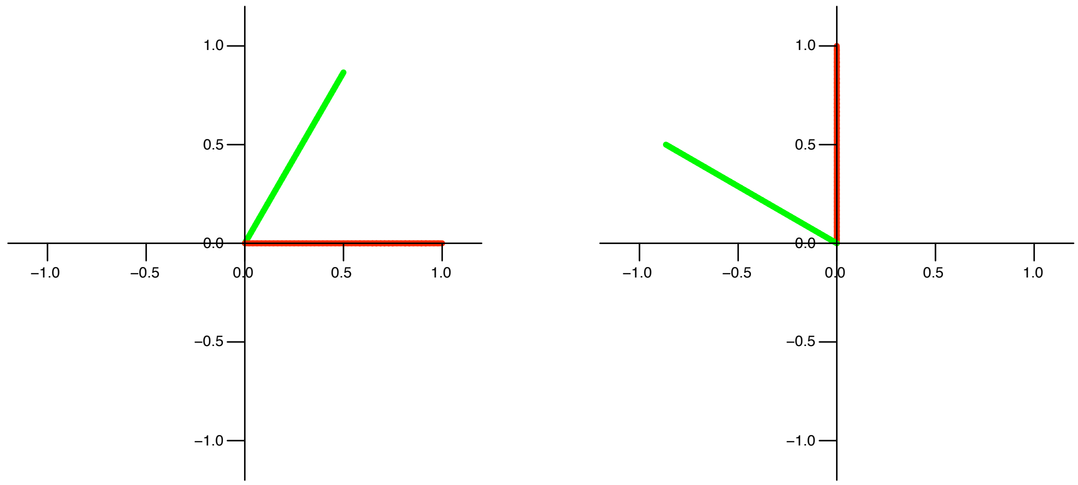
The uniqueness part of the result is a homework problem (#33). $\square$

The unique matrix A of Theorem 10 is called the **standard matrix for the linear transformation T**.

Example. Let T be the transformation on $\mathbb{R}^2$ that rotates each vector in the counterclockwise direction by θ radians.

It can be shown geometrically that this transformation is linear by looking at what it does to scalar multiples of vectors and to sums of vectors.

This linear transformation is not given by a formula, but by a picture. Using the definitions of sine and cosine, we can sketch the effect of T on the standard basis vectors.



The vectors $\vec{e}_1$ and $\vec{e}_2$ are the red lines segments (the horizontal one on the left, and the vertical one on the right), and the green lines (the nonhorizontal/nonvertical ones) are the vectors obtained by rotating $\vec{e}_1$ and $\vec{e}_2$ respectively counterclockwise by an angle θ radians.

Use trigonometry, the images of $\vec{e}_1$ and $\vec{e}_2$ under T are

$$T(\vec{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \ T(\vec{e}_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}.$$

The standard matrix representation for the linear transformation T is

$$A = \begin{bmatrix} T\vec{e}_1) & L(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

In particular when $\theta = \pi/2$ (rotation by 90°) we have

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

This says that the image of $\vec{e}_1$ under T is $\vec{e}_2$, and the image of $\vec{e}_2$ under T is $-\vec{e}_1$. / / / /

The text lists the standard matrices of several linear transformations of $\mathbb{R}^2$. We will not review them here. Please look at these yourselves.

We turn to properties that a transformation may or may not have, and characterize when a linear transformation has these properties.

Definition. A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **onto** (or surjective) if each $\vec{b}$ in $\mathbb{R}^m$ is the image of at least one $\vec{x}$ in $\mathbb{R}^n$, i.e., for all $\vec{b} \in \mathbb{R}^m$ there exists $\vec{x} \in \mathbb{R}^n$ such that $T(\vec{x}) = \vec{b}$.

In other words, for each $\vec{b}$ in $\mathbb{R}^m$, the equation $T(\vec{x}) = \vec{b}$ has at least one solution, i.e., the range of T is the codomain of T.

Definition. A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** (or injective) if each $b \in \mathbb{R}^m$ is the image under T of at most one $\vec{x}$ in $\mathbb{R}^n$.

In other words, for each $\vec{b}$ in $\mathbb{R}^m$, the equation $T(\vec{x}) = \vec{b}$ has either one solution (a unique solution) or no solution.

We can completely characterize when a linear transformation is one-to-one.

Theorem 11. Suppose a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear. Then T is one-to-one if and only if the equation $T(\vec{x}) = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$.

Proof. Since T is linear we know that $T(\vec{x}) = \vec{0}$ has the trivial solution $\vec{x} = \vec{0}$.

Suppose that T is one-to-one.

Then the equation $T(\vec{x}) = \vec{0}$ has at most one solution, and hence exactly one solution, namely the trivial solution.

To prove the implication "if the equation $T(\vec{x}) = \vec{0}$ has only the trivial solution, then T is one-to-one," we prove its contrapositive, namely, "if T is not one-to-one, then $T(\vec{x}) = \vec{0}$ has more than the trivial solution."

So we suppose that T is not one-to-one.

Then there exists $\vec{b}$ in $\mathbb{R}^m$ that is the image of two distinct $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, i.e., $T(\vec{u}) = \vec{b} = T(\vec{v})$ for $\vec{u} \neq \vec{v}$.

Then $\vec{u} - \vec{v} \neq \vec{0}$, so that by the linearity of T we obtain

$$T(\vec{u} - \vec{v}) = T(\vec{u}) - T(\vec{v}) = \vec{b} = \vec{b} = \vec{0}.$$

Thus the equation $T(\vec{x}) = \vec{0}$ has more than the trivial solution. □

The standard matrix A for a linear transformation T can be used to deduce when T is one–to-one and when it is onto.

Theorem 12. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and A its standard matrix.

a. T is onto if and only if the columns of A span $\mathbb{R}^m$.
b. T is one-to-one if and only if the columns of A are linearly independent.

Proof. (a) By Theorem 4, the columns of A span $\mathbb{R}^m$ if and only if for each $\vec{b}$ in $\mathbb{R}^m$ the equation $A\vec{x} = \vec{b}$ is consistent, that is, the equation $T(\vec{x}) = \vec{b}$ has at least one solution.

This is true if and only if T is onto.

(b) The equations $T(\vec{x}) = \vec{0}$ and $A\vec{x} = \vec{0}$ has the same solution set since $T(\vec{x}) = A\vec{x}$.

By Theorem 11, T is one-to-one if and only if $A\vec{x} = \vec{0}$ has only the trivial solution.

This happens if and only if the columns of A are linearly independent. □

Examples. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_2 - x_1 \\ x_1 - 3x_3 \end{bmatrix}.$$

Is this a linear transformation? Is it one-to-one? Is it onto?

Proceeding as though T were linear (we will verify this in a minute), the images of the vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ under T are

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\1\end{bmatrix}, \; T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\0\end{bmatrix}, \; T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\-3\end{bmatrix}.$$

We form the 2×3 matrix

$$A = \left[T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3)\right] = \begin{bmatrix}-1 & 2 & 0\\ 1 & 0 & -3\end{bmatrix}.$$

Then

$$A\vec{x} = \begin{bmatrix}-1 & 2 & 0\\ 1 & 0 & -3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}2x_2 - x_1\\ x_1 - 3x_3\end{bmatrix} = T(\vec{x}).$$

This shows that T is a matrix transformation, and hence linear.

Row-reducing A determines whether T is one-to-one (or not) and if T is onto (or not):

$$\begin{bmatrix}-1 & 2 & 0\\ 1 & 0 & -3\end{bmatrix} \; R_2 + R_1 \to R_2 \sim \begin{bmatrix}-1 & 2 & 0\\ 0 & 2 & -3\end{bmatrix}.$$

This says there is a free variable for the homogeneous equation $A\vec{x} = \vec{0}$, and so T is not one-to-one.

This also says the every row of A has a pivot position in it, so by Theorem 4, the columns of A span $\mathbb{R}^2$, and hence T is onto.