Math 313 Lecture #10
§2.2: The Inverse of a Matrix

Matrix algebra provides tools for creating many useful formulas just like real number algebra does.

For example, a real number a is invertible if there is a real number b such that

$$ba = ab = 1;$$

when this is the case, the real number b is called the multiplicative inverse of a, and of course, $b = a^{-1} = 1/a$. The only non invertible real number is 0.

An $n \times n$ matrix A is said to be **invertible** or **nonsingular** if there is an $n \times n$ matrix B such that

$$AB = BA = I;$$

otherwise, A is said to be **non-invertible** or **singular**.

When a square matrix A is invertible, the matrix B for which $AB = BA = I$ is called a **multiplicative inverse** of A.

The matrix B is unique (as you should have shown in the homework problem #25 in Section 2.1).

So, there is only one multiplicative inverse of an invertible matrix; the unique multiplicative inverse of an invertible matrix A is denoted by A^{-1}.

Example. Let $A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 7 & 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & -2 & -1 \\ -7 & -4 & -2 \\ -14 & -7 & -4 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} -4 & -2 & -1 \\ -7 & -4 & -2 \\ -14 & -7 & -4 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 7 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus A is invertible and $A^{-1} = B$. [We will learn latter in this lecture how to find A^{-1} when A is invertible, by row reduction.] □

When A is a 2×2 matrix, there is a "nice" formula for its inverse when it is invertible.

Theorem 4. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, if the quantity $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

We can verify that the given A^{-1} (when it exists, i.e., when $ad - bc \neq 0$) indeed satisfies $AA^{-1} = I$ and $A^{-1}A = I$.

The contrapositive of Theorem 4 is that if A is not invertible, then $ad - bc = 0$. [You have a homework problem #25 to prove that the converse – if $ad - bc = 0$, then A is not invertible – is true.]

Thus a 2×2 matrix is invertible if and only if $ad - bc \neq 0$; we call this number the **determinant** of A.

The existence of an inverse can be used to solve $A\vec{x} = \vec{b}$ when A is a square matrix.

Theorem 5. Suppose an $n \times n$ matrix A is invertible. Then for each $\vec{b} \in \mathbb{R}^n$ the equation $A\vec{x} = \vec{b}$ has a unique solution $\vec{x} = A^{-1}\vec{b}$.

Proof. Let $\vec{b}$ be in $\mathbb{R}^n$.

The "guess" $A^{-1}\vec{b}$ is a solution of $A\vec{x} = \vec{b}$ because

$$A(A^{-1}\vec{b}) = (AA^{-1})\vec{b} = I\vec{b} = \vec{b}.$$

To show that $A^{-1}\vec{b}$ is the only solution, we suppose that $\vec{u}$ is a solution of $A\vec{x} = \vec{b}$ and show that $\vec{u} = A^{-1}\vec{b}$.

To this end we multiply both sides of $A\vec{u} = \vec{b}$ by A^{-1} (on the left of each side) to get $A^{-1}(A\vec{u}) = A^{-1}\vec{b}$.

This simplifies to $\vec{u} = A^{-1}\vec{b}$, and so the solution is unique. $\square$

Notice in this proof that we used both $A^{-1}A = I$ and $AA^{-1} = I$.

Theorem 6 (Properties of an Inverse). Let A and B be $n \times n$ matrices.

a. If A is invertible, then A^{-1} is invertible and $(A^{-1})^{-1} = A$.
b. If A and B are invertible, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
c. If A is invertible, then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

Proof. (a) If A is invertible then A^{-1} exists, and $A^{-1}A = I$ and $AA^{-1} = I$, so that A^{-1} is invertible with inverse $(A^{-1})^{-1} = A$.

(b) If A and B are invertible, then $B^{-1}A^{-1}$ exists and $(AB)(B^{-1}A^{-1}) = AIA^{-1} = I$ and $(B^{-1}A^{-1})(AB) = B^{-1}IB = I$, so that AB is invertible with inverse $B^{-1}A^{-1}$.

(c) If A is invertible, then A^{-1} exists and $A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$ and $(A^{-1})^TA^T = (AA^{-1})^T = I^T = I$, so that A^T is invertible with inverse $(A^{-1})^T$. $\square$

Property (b) extends by induction so that the inverse of a product of a finite number of invertible matrices is the product of the inverse matrices in *reverse* order:

$$(A_1A_2 \cdots A_p)^{-1} = A_p^{-1} \cdots A_2^{-1}A_1^{-1}.$$

Elementary Matrices. The theoretical basis for finding the inverse of an invertible matrix is the use of invertible matrices "close to" the identity matrix.

A square matrix obtained by performing one elementary row operation on the identity matrix is called an **elementary matrix**.

Examples. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix},$$

and consider the following three matrices obtained from the identity matrix by a single elementary row operation:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} R_2 \leftrightarrow R_3 \sim \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix} = E_1,$$
$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} -3R_2 \to R_2 \sim \begin{bmatrix}1&0&0\\0&-3&0\\0&0&1\end{bmatrix} = E_2,$$
$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} R_3 + 2R_1 \to R_3 \sim \begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix} = E_3.$$

Then

$$E_1A = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix} = \begin{bmatrix}1&2&3\\7&8&9\\4&5&6\end{bmatrix},$$
$$E_2A = \begin{bmatrix}1&0&0\\0&-3&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix} = \begin{bmatrix}1&2&3\\-12&-15&-18\\7&8&9\end{bmatrix},$$
$$E_3A = \begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix} = \begin{bmatrix}1&2&3\\4&5&6\\9&12&15\end{bmatrix}.$$

Each row operation encodes as multiplication by a matrix on the left!

Which of the three elementary matrices are invertible? All of them are, where each inverse is another elementary matrix for the same type of row operation.

Theorem 7. An $n \times n$ matrix A is invertible if and only if A is row equivalent to I, i.e., there exists finitely many elementary matrices $E_1, E_2, \dots, E_p$ such that $E_p \cdots E_2E_1A = I$.

The proof of this is in the next lecture.

How can we use Theorem 7 to find the inverse of an invertible matrix?

Look at $E_p \cdots E_2E_1A = I$.

Multiplying both sides of this on the right by A^{-1} gives $A^{-1} = E_p \cdots E_2E_1$.

So if we augment A with I, written $\begin{bmatrix}A & | & I\end{bmatrix}$, and row reduce this augmented matrix by the row operations in the order $E_1, E_2, \dots, E_p$ we get

$$(E_p \cdots E_2E_1)\begin{bmatrix}A & | & I\end{bmatrix} \sim \begin{bmatrix}E_p \cdots E_2E_1A & | & E_p \cdots E_2E_1\end{bmatrix} = \begin{bmatrix}I & | & A^{-1}\end{bmatrix}.$$

We use row reduction to find the inverse of an invertible matrix A.

What would happen is A were not invertible and we row reduced $\begin{bmatrix}A & | & I\end{bmatrix}$?

Example. Find, if possible, the inverse of

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Augment A with I and row reduce:

$$\begin{aligned}
(A|I) = &\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{array} \\
\Leftrightarrow &\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + 2R_2 \to R_1 \\ \\ R_3 - R_2 \to R_3 \end{array} \\
\Leftrightarrow &\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 2 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right] \begin{array}{l} R_1 + 2R_3 \to R_1 \\ \\ \\ \end{array} \\
\Leftrightarrow &\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{array}\right] \begin{array}{l} \\ -R_2 \to R_2 \\ -R_3 \to R_3 \end{array} \\
\Leftrightarrow &\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right].
\end{aligned}$$

So

$$A^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

We can (and should) verify this:

$$AA^{-1} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A^{-1}A = I \text{ (this left to you)}\checkmark.$$

We can use A^{-1} to solve the system $A\vec{x} = \vec{b}$ for

$$\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

"Premultiplying" both sides of $A\vec{x} = \vec{b}$ by A^{-1}, i.e., $A^{-1}A\vec{x} = A^{-1}\vec{b}$, gives

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix}.$$

We can (and should) verify this:

$$A\vec{x} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \vec{b} \checkmark.$$

We would get the same solution if we row reduced $\left[A \quad | \quad \vec{b}\right]$? [Left for you to verify.]