Math 313 Lecture #12
§3.1: Introduction to Determinants

Recall that a 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is invertible if and only if the determinant

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

is nonzero.

To find the analogous formula for an 3×3 matrix (and eventually an $n \times n$ matrix), we make use of a invertible matrix being row equivalent to I.

If $a_{11} \neq 0$ in a 3×3 matrix $A = (a_{ij})$, then

$$\begin{aligned} A = &\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} \\ R_2 - (a_{21}/a_{11})R_1 \to R_2 \\ R_3 - (a_{31}/a_{11})R_1 \to R_3 \end{matrix} \\ \sim &\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} - a_{12}a_{21}/a_{11} & a_{23} - a_{13}a_{21}/a_{11} \\ 0 & a_{32} - a_{31}a_{12}/a_{11} & a_{33} - a_{31}a_{13}/a_{11} \end{bmatrix} \begin{matrix} \\ a_{11}R_2 \to R_2 \\ a_{11}R_3 \to R_3 \end{matrix} \\ \sim &\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{31}a_{13} \end{bmatrix}. \end{aligned}$$

This matrix is row equivalent to I if and only if the determinant of the lower right 2×2 submatrix is nonzero:

$$\begin{aligned} &\begin{vmatrix} a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{31}a_{13} \end{vmatrix} \\ &= a_{11}^2a_{22}a_{33} - a_{11}a_{13}a_{22}a_{31} - a_{11}a_{12}a_{21}a_{33} + a_{12}a_{13}a_{21}a_{31} \\ &\quad - \big(a_{11}^2a_{23}a_{32} - a_{11}a_{12}a_{23}a_{31} - a_{11}a_{13}a_{21}a_{32} + a_{12}a_{13}a_{21}a_{31}\big) \\ &= a_{11}^2a_{22}a_{33} - a_{11}a_{13}a_{22}a_{31} - a_{11}a_{12}a_{21}a_{33} \\ &\quad - a_{11}^2a_{23}a_{32} + a_{11}a_{12}a_{23}a_{31} + a_{11}a_{13}a_{21}a_{32} \\ &= a_{11}\big(a_{11}a_{22}a_{33} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}\big). \end{aligned}$$

Since $a_{11} \neq 0$, then A is invertible if and only if

$$a_{11}a_{22}a_{33} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \neq 0.$$

The **determinant** of a 3×3 matrix $A = (a_{ij})$ is defined by

$$\det(A) = a_{11}a_{22}a_{33} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}.$$

Now this is a terribly difficult expression to remember. Let's see if we can find a nemonic (i.e., a pattern) that is easier to remember but whose usage leads to the expression.

If we collect the a_{11} terms together, and the a_{12} terms together, and the a_{13} terms together, we get

$$\begin{aligned} &a_{11}\big(a_{22}a_{33} - a_{23}a_{32}\big) - a_{12}\big(a_{21}a_{33} - a_{23}a_{31}\big) + a_{13}\big(a_{21}a_{32} - a_{22}a_{31}\big) \\ &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}. \end{aligned}$$

Each of the 2×2 matrices appearing in this determinant are 2×2 submatrices of A!

Let A_{ij} be the 2×2 submatrix formed from A by deleting its i^{th} row and its j^{th} column.

Then the determinant of a 3×3 matrix $A = (a_{ij})$ is

$$\boxed{\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}).}$$

Cofactor Expansions of Matrices.

The use of determinants of 2×2 submatrices of a 3×3 matrix A to define (and eventually compute) the determinant of A generalizes to $n \times n$ matrices.

Let A be an $n \times n$ matrix, and let A_{ij} be the $(n-1) \times (n-1)$ matrix obtained from A by deleting its i^{th} row and its j^{th} column.

The **cofactor** of a_{ij} is defined to be the quantity

$$C_{ij} = (-1)^{i+j}\det(A_{ij}).$$

In terms of cofactors, the determinant of a 3×3 matrix $A = (a_{ij})$ takes the form

$$\begin{aligned} \det(A) &= a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) \\ &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}. \end{aligned}$$

This is the form of the determinant that generalizes to an $n \times n$ matrix.

The **determinant** of an $n \times n$ matrix $A = (a_{ij})$ is the quantity defined by

$$\boxed{\det(A) = \sum_{k=1}^{n} a_{1k}C_{1k} = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.}$$

This nemonic is called the **cofactor expansion of** A **across its first row**. [Notice that $a_{11}, a_{12}, \ldots, a_{1n}$ are the entries of the first row of A.]

This nemonic computes the determinant of a 4×4 matrix A as a sum of scalar multiples of determinants of four 3×3 submatrices of A.

By induction the nemonic bootstraps to any size of a square matrix.

Another notation for the determinant of A if $|A|$.

Example. Find the determinant of $A = \begin{bmatrix} 2 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 & 0 \end{bmatrix}$.

Apply cofactor expansion along the first row over and over again until we get to 2×2 matrices whose determinants are easy to compute:

$$\begin{aligned}
\begin{vmatrix} 2 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 & 0 \end{vmatrix} &= 2 \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 1 & 0 \end{vmatrix} - 3 \begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 1 & 0 \end{vmatrix} \\
&= 2 \left(-(1) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{vmatrix} \right) - 3 \left((1) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 0 \end{vmatrix} - (1) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{vmatrix} \right) \\
&= -2(1) \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 3(1) \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + 3(1) \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \\
&= -2(1-2) - 3(1-2) + 3(1-2) \\
&= 2.
\end{aligned}$$

Is cofactor expansion across the first row the only way to compute determinants?

Theorem 1. The determinant of an $n \times n$ matrix A can be computed by cofactor expansion across *any* row or across *any* column:

$$\boxed{\det(A) = \sum_{k=1}^{n} a_{ik} C_{ik} = \sum_{k=1}^{n} a_{kj} C_{kj}}$$

for all $i = 1, \dots, n$ and all $j = 1, \dots, n$.

The proof of this involves much messy algebra.

There are matrices for which the determinants are easy to compute.

Theorem 2. If A is a triangular $n \times n$ matrix, then $\det(A) = a_{11}a_{22} \cdots a_{nn}$ (the product of the entries of the main diagonal).

Proof. Cofactor expanding across the first column gives the determinant as the $(1,1)$ entry of A times the determinant of the $(n-1) \times (n-1)$ matrix A_{11}.

Now repeat this with A_{11} in place of A, until we arrive at a 2×2 matrix for which determinant is the product of the entries of the main diagonal by the 2×2 determinant formula. □

Example. $\begin{vmatrix} 3 & 100 & -19 \\ 0 & 2 & 56 \\ 0 & 0 & 4 \end{vmatrix} = (3)(2)(4) = 24.$