# Math 313 Lecture #13
## §3.2: Properties of Determinants

It would be great if we could first row reduce an $n \times n$ matrix $A$ and then compute its determinant from the simpler row reduced matrix.

But alas, this method is *doomed to fail*: an invertible matrix $A$ is row equivalent to $I$ and the determinant of $I$ (a diagonal matrix with 1's on the main diagonal) is 1, but is $\det(A) = \det(I)$?

But not all is lost: there is a way to recover the determinant of $A$ from an echelon form **AND** the row operations needed to get to that echelon form.

Theorem 3. Let $A$ be a square matrix.

a. If $B$ is obtained from $A$ by adding a multiple of one row of $A$ to another row of $A$, then $\det(B) = \det(A)$.
b. If $B$ is obtained from $A$ by switching two rows of $A$, then $\det(B) = -\det(A)$.
c. If $B$ is obtained from $A$ by multiplying a row of $A$ through by a scalar $k$, then $\det(B) = k\det(A)$.

Idea of Proof. We use induction, where the base case is for a $2 \times 2$ matrix $A$.

You had homework that verified what some of the row operations did to the determinant of a $2 \times 2$ matrix.

The effect of the remaining row operations on a $2 \times 2$ matrix are similarly verified.

Now assume that the statements of the Theorem hold for a $k \times k$ matrix $A$ with $k \geq 2$.

The effect of an elementary matrix $E$ on $A$ leaves at least one row of $A$ in the same place without changes in its entries, say the $i^{\text{th}}$ row.

Cofactor expansion of $B = EA$ across the $i^{\text{th}}$ row results in

$$\det(EA) = (-1)^{i+1}a_{i1}\det(B_{1i}) + (-1)^{i+2}a_{i2}\det(B_{2i}) + \cdots + (-1)^{i+n}a_{in}\det(B_{in}).$$

We recognize that the rows of each $k \times k$ matrix $B_{ij}$ are obtained from the rows of each $k \times k$ matrix $A_{ij}$ by the same type elementary row operations as $E$.

By the induction hypothesis we have $\det(B_{ij}) = \alpha\det(A_{ij})$ for all $i = 1, 2, \ldots, n$ where $\alpha$ is 1 if $E$ is scalar multiple of one row added to another, $-1$ if $E$ is a switching of two rows, and $r$ if $E$ is scalar multiplying a row through by $r$.

We obtain

$$\det(EA) = \alpha\big[(-1)^{i+1}a_{i1}\det(A_{1i}) + (-1)^{i+2}a_{i2}\det(A_{2i}) + \cdots + (-1)^{i+n}a_{in}\det(A_{in})\big]$$

which says that $\det(EA) = \det(E)\det(A)$. $\square$

We can use row reduction of a square matrix to find its determinant as long as we keep track of the row operations that change the determinant.

Suppose an echelon form $U$ is obtained from $A$ by using only the row operations of switching rows and adding multiples of one row to another row (which can always be done without the third row operation).

If $r$ is the number of row switches performed, then $\det(A)$ is $(-1)^r\det(U)$.

The determinant of $U$ is just the product of diagonal entries of $U$ because $U$ is in triangular form.

**Example.** Use row reduction to find the determinant:

$$\begin{vmatrix} 2 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 & 0 \end{vmatrix} \overset{R_1 \leftrightarrow R_2}{} = (-1) \begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 2 & 3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 & 0 \end{vmatrix} \begin{matrix} R_2 - 2R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{matrix}$$

$$= -(1)(1) \begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 & 0 \end{vmatrix} \overset{R_2 \leftrightarrow R_3}{} = -(-1) \begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 & 0 \end{vmatrix} \begin{matrix} R_3 - 3R_2 \to R_3 \\ \\ R_5 - 2R_4 \to R_4 \end{matrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -1 & -4 \end{vmatrix} \overset{R_3 \leftrightarrow R_4}{} = (-1) \begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 & -4 \end{vmatrix} R_4 \leftrightarrow R_5$$

$$= -(-1) \begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 0 & -2 \end{vmatrix} = 2.$$

Notice that we did not have to use the row operation of multiplying a row through by a nonzero scalar. ////

We saw before that a $2 \times 2$ matrix $A$ is invertible if and only if $\det(A) \neq 0$.

This is also true for any $n \times n$ matrix, and adds another statement to the Inverse Matrix Theorem.

**Theorem 4.** A square matrix $A$ is invertible if and only if $\det(A) \neq 0$.

Proof. Suppose $A$ is invertible.

By the Inverse Matrix Theorem, $A$ has $n$ pivot positions, and these lie on the main diagonal of $A$.

Hence for an echelon form $U$ for $A$, obtained by using only $r$ row exchanges and multiples of one row added to another, we have $\det(A) = (-1)^r\det(U)$ where none of the main diagonal entries of $U$ are zero.

This implies that $\det(A) \neq 0$.

Now suppose that $\det(A) \neq 0$.

Then row reducing $A$ to an echelon form $U$, using on $r$ row switches and multiples of one row added to another row, we have $\det(A) = (-1)^r \det(U)$.

With $\det(A) \neq 0$, we have that none of the diagonal entries of $U$ can be zero (as their product is the determinant).

Thus every diagonal entry of $U$ is pivot position for $A$.

By the Inverse Matrix Theorem, the matrix $A$ is invertible. □

What does the transpose do to the determinant of a square matrix?

Theorem 5. For an $n \times n$ matrix, we have $\det(A^T) = \det(A)$.

Idea of Proof. We use induction with $n = 2$ as the base case.

By the formula for the determinant of a $2 \times 2$ matrix $A$, we immediately get $\det(A^T) = \det(A)$.

Now suppose for any $k \times k$ matrix that transposition does not change the determinant.

Then the cofactor of $a_{1j}$ in a $(k+1) \times (k+1)$ matrix $A$ equals the cofactor of $a_{j1}$ in $A^T$ by the induction hypothesis because the cofactors involve $k \times k$ determinants.

Hence cofactor expansion across the first row of $A$ equals the cofactor expansion down the first column of $A^T$. □

Recall for an elementary matrix $E$ that $\det(EA) = \det(E)\det(A)$.

It would be great if this works when $E$ is not an elementary matrix. And it is great that it does!

Theorem 6. If $A$ and $B$ are $n \times n$ matrices, then

$$\det(AB) = \det(A)\det(B).$$

Proof. If $A$ is singular, then $AB$ is singular too (why?), so that

$$\det(AB) = 0 = \det(A)\det(B).$$

If $A$ is nonsingular, then $A$ is the product of elementary matrices $E_k, \ldots, E_1$.

Since $\det(EC) = \det(E)\det(C)$ for any elementary matrix $E$ and any matrix $C$, then

$$\begin{aligned}\det(AB) &= \det\big(E_k \cdots E_1 B\big)\\ &= \det(E_k)\det(E_{k-1} \cdots E_2 E_1 B)\\ &= \det(E_k) \cdots \det(E_1)\dim(B)\\ &= \det(E_k \cdots E_1)\det(B)\\ &= \det(A)\det(B).\end{aligned}$$

This completes the proof. □