# Math 313 Lecture #15
## §4.1: Vector Spaces and Subspaces

**Vector Spaces.** Many sets of mathematical objects have the algebraic properties that $\mathbb{R}^n$ does.

They have a scalar multiplication and an addition that behave just like those of $\mathbb{R}^n$.

The idea of a vector space is a unifying principle in that properties of the vectors depend only on the properties of the scalar multiplication and addition but not on the elements of the vector space.

**Definition.** A **vector space** is a set $V$ of objects denoted as $\vec{u}$, $\vec{v}$, $\vec{w}$, etc., on which are defined two operations, scalar multiplication $\alpha\vec{u}$ and addition $\vec{u}+\vec{v}$, which operations satisfy the following axioms:

1. the sum $\vec{u}+\vec{v}$ is in $V$ for all $\vec{u}, \vec{v}$ in $V$,
2. $\vec{x}+\vec{y} = \vec{y}+\vec{x}$ for all $\vec{x}, \vec{y} \in V$,
3. $(\vec{x}+\vec{y})+\vec{w} = \vec{x}+(\vec{y}+\vec{w})$ for all $\vec{x}, \vec{y}, \vec{w} \in V$,
4. There exists an element $\vec{0}$ in $V$ for which $\vec{x}+\vec{0} = \vec{x}$ for each $\vec{x} \in V$,
5. For each $\vec{x} \in V$ there is an element $-\vec{x}$ such that $\vec{x}+(-\vec{x}) = \vec{0}$,
6. the scalar multiple $\alpha\vec{u}$ is in $V$ for all $\alpha$ and for all $\vec{u}$ in $V$,
7. $\alpha(\vec{x}+\vec{y}) = \alpha\vec{x}+\alpha\vec{y}$ for all $\alpha \in \mathbb{R}$ and all $\vec{x}, \vec{y} \in V$,
8. $(\alpha+\beta)\vec{x} = \alpha\vec{x}+\beta\vec{x}$ for all $\alpha, \beta \in \mathbb{R}$ and all $\vec{x} \in V$,
9. $(\alpha\beta)\vec{x} = \alpha(\beta\vec{x})$ for all $\alpha\beta \in \mathbb{R}$ and all $\vec{x} \in V$,
10. $1\vec{x} = \vec{x}$ for all $\vec{x} \in V$.

Using ONLY these properties, we can show that the vector $\vec{0}$ of Axiom 4 is unique, that the negative $-\vec{u}$ of Axiom 5 is unique, that $0\vec{u} = \vec{0}$, that $\alpha\vec{0} = \vec{0}$, and that $-\vec{u} = (-1)\vec{u}$.

**The Vector Space $\mathbb{R}^{m\times n}$.** The set of all $m \times n$ matrices is denoted by $\mathbb{R}^{m\times n}$.

The set $\mathbb{R}^{m\times n}$ has a scalar multiplication and addition defined on it: if $A = (a_{ij})$, $B = (b_{ij})$ are $m \times n$ matrices, and $\alpha$ is a scalar, then

$$\alpha A = (\alpha a_{ij}) \quad \text{and} \quad A + B = (a_{ij} + b_{ij}).$$

The algebraic operations of scalar multiplication and of addition obey the ten axioms of a vector space: these rules are nothing more than the rules of matrix algebra.

**The Vector Space $C[a, b]$.** Now for a set of "objects" you may not have thought of as a vector space.

Let $[a, b]$ be a closed interval with finite nonzero length, i.e., $-\infty < a < b < \infty$.

Let $C[a, b]$ denote the set of functions continuous on $[a, b]$ (where continuity at the end-points is understood as one-sided).

On $C[a,b]$ there is a scalar multiplication: if $\alpha \in \mathbb{R}$ and $f \in C[a,b]$, then the scalar multiple of $f$ by $\alpha$ is the continuous function

$$(\alpha f)(x) = \alpha f(x),\ x \in [a,b].$$

On $C[a,b]$ there is an addition: if $f$ and $g$ are in $C[a,b]$, then the sum of $f$ and $g$ is the continuous function

$$(f+g)(x) = f(x) + g(x),\ x \in [a,b].$$

Axioms 2 and 3 are satisfied here:

$$\begin{aligned}
&(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x), \\
&\big((f+g)+h\big)(x) = (f+g)(x) + h(x) = f(x) + g(x) + h(x) \\
&= f(x) + (g(x) + h(x)) = \big(f + (g+h)\big)(x).
\end{aligned}$$

What is the zero vector in $C[a,b]$? It is the zero function,

$$z(x) = 0 \text{ for all } z \in [a,b].$$

It satisfies $(f+z)(x) = f(x)$. Thus axiom 4 is satisfied.

The rest of the axioms are left to you to verify.

**The Vector Space $\mathbb{P}_n$.** Let $\mathbb{P}_n$ denote the set of all polynomials of degree at most $n$, having real coefficients.

Scalar multiplication and addition on $\mathbb{P}_n$ are defined like those on $C[a,b]$:

$$(\alpha p)(x) = \alpha p(x), \quad (p+q)(x) = p(x) + q(x).$$

The scalar multiple of a polynomial of degree at most $n$, with real coefficients, is a polynomial of degree at most $n$, with real coefficients.

The addition of two polynomials of degree at most $n$, with real coefficients, is a polynomial of degree at most $n$, with real coefficients.

Thus these algebraic operations on $\mathbb{P}_n$ satisfy axioms 1 and 6.

What is the zero vector in $\mathbb{P}_n$? It is the zero polynomial,

$$z(x) = 0x^n + \cdots + 0x + 0.$$

It is left to you to verify that the other axioms are satisfied.

**Subspaces.** Certain subsets of a vector space $V$ are themselves vector spaces.

To detect which subsets $H$ of $V$ are subspaces requires verifying only three of the axioms in $H$, because the rest following from the axioms of $V$.

**Definition.** A **subspace** of a vector space $V$ is a subset $H$ for which

a. $H$ contains the zero vector $\vec{0}$ of $V$,
b. $H$ is closed under addition, i.e., for every $\vec{u}$ and $\vec{w}$ are in $H$ we have $\vec{u} + \vec{v}$ is in $H$,
c. $H$ is closed under scalar multiplication, i.e., for every $\vec{u}$ in $H$ and every scalar $\alpha$, we have $\alpha\vec{u}$ is in $H$.

Example. Let $S = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2} : a_{11} + a_{22} = 0 \right\}$.

We check the three conditions in the definition of a subspace.

The zero $2 \times 2$ matrix 0 satisfies $a_{11} + a_{22} = 0$, and so $0 \in S$.

If $\alpha \in \mathbb{R}$, $A = (a_{ij}) \in S$, and $B = (b_{ij}) \in S$, then

$$\alpha A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{bmatrix}$$

satisfies $\alpha a_{11} + \alpha a_{22} = \alpha(a_{11} + a_{22}) = 0$, and

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

satisfies $(a_{11} + b_{11}) + (a_{22} + b_{22}) = (a_{11} + a_{22}) + (b_{11} + b_{22}) = 0$.

Thus $S$ is a subspace of $\mathbb{R}^{2\times 2}$.

Example. Let $\mathbb{P}$ be the set of all polynomials with real coefficients.

The subset $\mathbb{P}_n$ of polynomials of degree at most n is a subspace of $\mathbb{P}$.

Example. Is the subset

$$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

a subspace of $\mathbb{R}^3$?

Well, $H$ contains $\vec{0}$, scalar multiples of elements of $H$ are in $H$, and the sum of any two vectors in $H$ is in $H$, and so $H$ is a subspace of $\mathbb{R}^3$.

Is $H$ just $\mathbb{R}^2$? It isn't. Why not?

Example. Let $L$ be a line in the plane $\mathbb{R}^2$ that does not pass through the origin.

Is $L$ a subspace of $\mathbb{R}^2$? No, why not? It does not contain $\vec{0}$, scalar multiples of vectors in $L$ are not in $L$, and sums of vectors in $L$ are not in $L$.

Example. Let $S = \{p(x) \in \mathbb{P}_5 : p(1) + p'(1) = 0\}$, the subset of those polynomials of degree at most 5 for which $p(1)$ plus its derivative $p'(1)$ equals 0.

The zero polynomial $z(x)$ satisfies $z(1) + z'(1) = 0$, and so $z(x) \in S$.

If $\alpha \in \mathbb{R}$, $p(x) \in S$, and $q(x) \in S$, then $\alpha p(x)$ satisfies

$$\alpha p(1) + \alpha p'(1) = \alpha\big(p(1) + p'(1)\big) = 0,$$

and $(p + q)(x)$ satisfies

$$\big(p(1) + q(1)\big) + \big(p'(1) + q'(1)\big) = \big(p(1) + p'(1)\big) + \big(q(1) + q'(1)\big) = 0.$$

Thus the subset $S$ is a subspace of $\mathbb{P}_5$.

A Subspace Spanned by a Set. A common way of describing a subspace is by the means of linear combinations.

Theorem 1. Let $V$ be a vector space and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ vectors in $V$. Then the set $H = \text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a subspace of $V$.

Proof. The "trivial" linear combination

$$0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_k = \vec{0} \in H.$$

For any scalar $\alpha$ and any $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k \in H$, we have

$$\alpha\vec{v} = (\alpha c_1)\vec{v}_1 + (\alpha c_2)\vec{v}_2 + \cdots + (\alpha c_k)\vec{v}_k \in H,$$

For two vectors $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$ and $\vec{u} = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_k\vec{v}_k$ we have

$$\vec{v} + \vec{u} = (c_1 + d_1)\vec{v}_1 + (c_2 + d_2)\vec{v}_2 + \cdots + (c_k + d_k)\vec{v}_k \in H.$$

Thus $H$ is a subspace of $V$. □

[Notice how these calculations have nothing to do with what the actual vectors in $V$ are, but only with the algebraic properties of scalar multiplication and addition.]

We call $\text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ the **subspace spanned** (or **generated**) by $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$.

A **spanning** (or **generating**) set for a subspace $H$ of $V$ is a set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ for which $H = \text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$.

Example. For an $m \times n$ matrix $A$, suppose that the solution set $H$ of $A\vec{x} = \vec{0}$ has the form $\vec{x} = s\vec{a} + t\vec{b}$.

Then the solution set $H$ is the subspace of $\mathbb{R}^n$ spanned by $\{\vec{a}, \vec{b}\}$.