Math 313 Lecture #16
§4.2: Null Spaces, Columns Spaces, and Linear Transformations

There are several subspaces associated with an $m \times n$ matrix A that are useful in analyzing A.

Having used these subspaces, we give them names.

Definition. The **null space** of an $m \times n$ matrix A is the set of all the solutions of the homogeneous equation $A\vec{x} = \vec{0}$, which we denote by

$$\text{Nul}(A) = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}.$$

Theorem 2. The null space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$.

Proof. We have three conditions to verify about the set $\text{Nul}(A)$.

First, does $\text{Nul}(A)$ contain $\vec{0}$? Yes, because $A\vec{0} = \vec{0}$.

Second, if $\vec{x} \in \text{Nul}(A)$, does $\alpha\vec{x}$ belong to $\text{Nul}(A)$ for all α? Yes, because

$$A(\alpha\vec{x}) = \alpha(A\vec{x}) = \alpha\vec{0} = \vec{0}.$$

Last, if $\vec{x}$ and $\vec{y}$ belong to $\text{Nul}(A)$, does $\vec{x} + \vec{y}$ belong to $\text{Nul}(A)$? Yes, because

$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0}.$$

Thus $\text{Nul}(A)$ is a subspace of $\mathbb{R}^n$. $\square$

Example. Find the null space of

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & -1 & 3 & 4 \\ 3 & 1 & 2 & 6 \end{bmatrix}.$$

The reduced row echelon form of A is

$$A \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The variables x_1 and x_2 are basic, and the variables x_3 and x_4 are free:

$$x_1 = -x_3 - 2x_4,$$
$$x_2 = x_3.$$

Let $x_3 = t$ and $x_4 = s$. The solution set of $A\vec{x} = \vec{0}$ is

$$\text{Nul}(A) = \left\{ \begin{bmatrix} -t - 2s \\ t \\ t \\ s \end{bmatrix} : t, s \in \mathbb{R} \right\} = \left\{ t \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} : t, s \in \mathbb{R} \right\}.$$

The null space of A is the span of two vectors in $\mathbb{R}^4$.

Are these two vector linearly independent? Yes, they are, because of different entries where the 0's and the 1's appear. / / / /

This pattern for the null space of a matrix holds whenever there are 1 or more free variables: we get a linearly independent set of vectors whose span is the null space, and the number of vectors in the spanning set is the number of free variables.

The Column Space of a Matrix. Another subspace associated with a matrix is its column space.

Definition. For an $m \times n$ matrix A , the **column space** of A is the subset of all linear combinations of the columns of A.

Theorem 3. The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Proof. We recognize that for $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$ we have

$$\text{Col}(A) = \text{Span}\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\},$$

and so it follows that $\text{Col}(A)$ is subspace of $\mathbb{R}^m$. □

Other ways to view the column space of a matrix are as

$$\text{Col})(A) = \{\vec{b} \in \mathbb{R}^n : A\vec{x} = \vec{b} \text{ for some } \vec{x} \in \mathbb{R}^n\}.$$

or as the range of the linear transformation $\vec{x} \mapsto A\vec{x}$.

Comparing and Contrasting the Null Space and Column Space of a Matrix. We have now associated to an $m \times n$ matrix A the two subspaces $\text{Nul}(A)$ and $\text{Col}(A)$.

Which one is a subspace of the domain $\mathbb{R}^n$? ($\text{Nul}(A)$) of the codomain $\mathbb{R}^m$? ($\text{Col}(A)$)

Which one is easier to compute? The column space is because it is the span of the columns of A, while to find the null space we have to row reduce A.

When does $\text{Nul}(A) = \{\vec{0}\}$ (the trivial subspace of $\mathbb{R}^n$)? When $A\vec{x} = \vec{0}$ has only the trivial solution.

When does $\text{Col}(A) = \mathbb{R}^m$? When $A\vec{x} = \vec{b}$ is consistent for all $\vec{b} \in \mathbb{R}^n$.

There are more of these comparisons and contrasts in the text.

Generalization of Linear Transformations. Previous for a linear transformation, we set its domain to be the vector space $\mathbb{R}^n$ and its codomain to be the vector space $\mathbb{R}^m$.

We can generalize the definition of the linear transformation to a domain that is a vector space and a codomain that is a vector space.

Definition. Let V and W be vector spaces. A transformation $T : V \to W$ is a rule that assigns to each $\vec{v}$ in V a vector $T(\vec{v})$ in W. A transformation $T : V \to W$ is linear if

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v} \in V$, and
2. $T(\alpha\vec{v}) = \alpha T(\vec{v})$ for all $\alpha \in \mathbb{R}$ and all $\vec{v} \in V$.

We can also extend the notions of null space and column space of a linear transformation $\vec{x} \mapsto A\vec{x}$ to a linear transformation $T : V \to W$.

Definition. The **kernel** of a linear transformation is the set of all $\vec{v}$ in V such that $T(\vec{v}) = \vec{0}$.

Is the kernel of a linear transformation a subspace of V?

Yes it is because $T(\vec{0}) = T(0\vec{0}) = 0T(\vec{0}) = \vec{0}$, and for $\vec{u}$ and $\vec{v}$ in the kernel of a linear T we have

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) = \vec{0} + \vec{=}\vec{0}, \quad T(\alpha\vec{u}) = \alpha T(\vec{u}) = \alpha\vec{0} = \vec{0}.$$

Recall the **range** of a transformation $T : V \to W$ (not assumed linear) is the set of vectors $\vec{w}$ in W for which there exists $\vec{v}$ in V such that $T(\vec{v}) = \vec{w}$.

If T is linear, if the range of T a subspace of W?

Yes it is because for $\vec{w}$ and $\vec{z}$ in the range of a linear T there are $\vec{u}$ and $\vec{v}$ in V for which $T(\vec{u}) = \vec{w}$ and $T(\vec{v}) = \vec{w}$, so that

$$\vec{w} + \vec{z} = T(\vec{u}) + T(\vec{v}) = T(\vec{u} + \vec{v}), \quad \alpha\vec{w} = \alpha T(\vec{u}) = T(\alpha\vec{u}),$$

saying that $\vec{w} + \vec{z}$ and $\alpha\vec{w}$ are in the range of T.

For the range to be subspace, the zero vector needs to be in it. Is $\vec{0}$ in the range of a linear T?

Yes it is because $T(\vec{0}) = T(0\vec{0}) = 0T(\vec{0}) = \vec{0}$.

Example Let $V = C^1[a, b]$ be the vector space of all real valued function defined on $[a, b]$ such that they are differentiable with continuous derivatives.

Let $W = C[a, b]$, the vector space of continuous real valued functions defined on $[a, b]$.

Is the transformation $D : V \to W$ defined by $D(f) = f'$ linear?

For f and g in V we have

$$D(f + g) = (f + g)' = f' + g', \quad D(\alpha f)(\alpha f)' = \alpha f',$$

so indeed D is linear.

What is the kernel of D? The subspace of all of the constant functions.

What is the range? It is all of W by the Fundamental Theorem of Calculus which says that any continuous function h on $[a, b]$ is the derivative of a function with a continuous derivative.