Math 313 Lecture #17
§4.3: Linearly Independent Sets; Bases

Having a spanning set of a subspace of a vector space enables us to describe all of the vectors in the subspace in terms of linear combination of the spanning set: know a few vectors, know all the vectors.

Not all spanning sets are of the same value: some are far superior to others.

We will learn today how to find the best possible spanning sets for subspaces.

To do this, we generalize (and review) the definitions of linear independence and linear dependence from vectors in $\mathbb{R}^n$ to vectors in a vector space V.

Defintion. A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ in a vector space V is linearly independent if the vector equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0}$$

has only the trivial solution $c_1 = c_2 = \cdots = c_p = 0$.

Otherwise the set of vectors is linearly dependent, i.e., we can satisfy the vector equation with at least some $c_i \neq 0$ (a nontrivial solution).

Before we learned to detect linear independence by row reducing a matrix whose columns were the elements of the given set of vectors in $\mathbb{R}^n$.

For an arbitrary vector space V, the vectors are not column vectors in $\mathbb{R}^n$, so that we have to work with the above vector equation directly.

But we still have tools (theorems) to help us identify linearly dependent or linearly independent sets.

Theorem 4. A set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ of two or more vectors in a vector space V with $\vec{v}_1 \neq \vec{0}$, are linearly dependent if and only if there is some $\vec{v}_j$ (with $j \geq 2$) such that v_j is a linear combination of the preceding vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{j-1}$.

The proof of this is hopefully by now not that difficult, and will be skipped.

Example. Is the set of polynomials $\{2 + t, 1 - t, 1 + t\}$ in $\mathbb{P}_1$ linearly independent?

Label the polynomials $p_1(t) = 2 + t$, $p_2(t) = 1 - t$, and $p_3(t) = 1 + t$.

Notice that

$$\frac{2p_1}{3} - \frac{p_2}{3} = \frac{4}{3} + \frac{2t}{3} - \frac{1}{3} + \frac{t}{3} = 1 + t = p_3$$

Since p_3 is a linear combination of p_1 and p_2, the set $\{p_1, p_2, p_3\}$ is linearly dependent by Theorem 4.

However the set $\{p_1, p_2\}$ is linearly independent because p_1 is not a scalar multiple of p_2, nor is p_2 a scalar multiple of p_1.

Does the linearly independent set $\{p_1, p_2\}$ span $\mathbb{P}_1$? Is every polynomial p in $\mathbb{P}_1$ a linear combination of the element of the linearly independent set $\{p_1, p_2\}$?

The answer is yes. ////

This finding of a linearly independent set in a vector space (or subspace) that spans the vector space (or subspace) illustrates the notion of best possible spanning set, and we give is a proper name.

Definition. Let H be a subspace of a vector space. A set of vectors $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_p\}$ in V is a **basis** for H if

(i) $\mathcal{B}$ is a linearly independent set, and
(ii) $\mathcal{B}$ spans H.

Since V is a subspace of V, this definition also applies to V, so that a basis of V is a linearly independent set of vectors in V that span V.

Example. A basis for the vector space $\mathbb{R}^n$ are the columns of the $n \times n$ identity matrix, namely,

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \ \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

This basis of $\mathbb{R}^n$ is called the **standard basis** of $\mathbb{R}^n$.

Example. What is a basis of $\mathbb{P}_n$?

Every polynomial $p(t)$ in $\mathbb{P}_n$ has the form

$$p(t) = c_0 1 + c_1 t + c_2 t^2 + \cdots + c_n t^n.$$

We recognize in this a linear combination of the polynomials $1, t, \ldots, t^n$.

This means that $\mathbb{P}_n$ is spanned by $1, t, \ldots, t^n$.

Is the spanning set $\{1, t, \ldots, t^n\}$ linearly independent?

We answer this by setting $c_0 1 + c_1 t + \cdots + c_n t^n = 0$ (this equal holds for all $t \in \mathbb{R}$) and determining what the weights c_i can be.

This says that each value of t is a root of the polynomial; it has infinitely many roots.

The Fundamental Theorem of Algebra states that a nonzero polynomial has at most n real roots.

Could $c_0 1 + c_1 t + \cdots c_n t^n$ be a nonzero polynomial with infinitely many roots?

NO, so the only way for $c_0 1 + c_1 t + \cdots + c_n t^n = 0$ to hold for all $t \in \mathbb{R}$) is that all of the weights are 0.

Hence $\{1, t, \ldots, t^n\}$ is a linearly independent set, and so it is a basis of $\mathbb{P}_n$. $////$

What kind of algorithm is there for finding a basis of a vector space (or any of its subspaces)?

Theorem 5 (The Spanning Set Theorem). Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ be a set in a vector space V, and set $H = \mathrm{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$.

a. If one of the vectors in S, say $\vec{v}_k$, is a linear combination of the other $p-1$ vectors in S, then the set obtained by removing $\vec{v}_k$ from S still spans H.
b. If $H \neq \{\vec{0}\}$, then some subset of H is a basis for H.

Proof. (a) We may assume WLOG (by relabelling the vectors in S) that

$$\vec{v}_p = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_{p-1}\vec{v}_{p-1}.$$

For an arbitrary $\vec{x}$ in H we then have that

$$\begin{aligned}\vec{x} &= c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_{p-1}\vec{v}_{p-1} + c_p\vec{v}_p \\ &= c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_{p-1}\vec{v}_{p-1} + c_p\big(\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_{p-1}\vec{v}_{p-1}\big) \\ &= (c_1 + \alpha_1 c_p)\vec{v}_1 + (c_2 + \alpha_2 c_p)\vec{v}_2 + \cdots + (c_{p-1} + \alpha_{p-1}c_p)\vec{v}_{p-1}.\end{aligned}$$

This says that the arbitrary $\vec{x}$ is in the span of the set obtained from S by removing $\vec{v}_p$.

(b) If the original spanning set S is linearly independent, then we have a basis for H.

So we may suppose that the spanning set S is not linearly independent.

Then one of the vectors in S, say WLOG the last one $\vec{v}_p$, is a linear combination of the others.

Removing $\vec{v}_p$ from S results in a new set, still call it S, that spans H by part (a).

If the new spanning set S is linearly independent we have a basis.

If not, then one the vectors, say WLOG $\vec{v}_{p-1}$, is a linear combination of the others, and can be removed form S, still call it S, that spans H.

We continue removing vectors from S until we arrive a linear independent set that spans H.

The process either ends with two linearly independent vectors in S that span H, or one vector (which has to be nonzero because $H \neq \{\vec{0}\}$), and so is linearly independent. □

Bases for Null Spaces and Columns Spaces. Recall that we find a spanning set for the null space of a matrix A by solving the homogeneous equation $A\vec{x} = \vec{0}$ in parametric vector form.

Also recall that the spanning set is linearly independent because of where the 0's and the 1's appear, and so we obtain a basis for $\mathrm{Nul}(A)$.

In finding a basis for the column space of A, we always have a spanning set, not necessarily a basis, given by the columns of A.

If the columns of A are linearly dependent, then by the Spanning Set Theorem, we can discard some columns and obtain a basis for $\mathrm{Col}(A)$.

How do we decide which columns of A to discard?

If you say by row reduction of A, you are correct.

Theorem 6. The pivot columns of A form a basis for $\mathrm{Col}(A)$.

Proof. Let U be the reduced row echelon form of A.

The set of pivot columns of U is linearly independent because a given pivot column cannot be a linear combination of the pivot columns before it (think of where the 1's are).

The pivot columns of A are linearly independent as well because any linear dependence relation among the columns of A (a solution $\vec{x}$ of $A\vec{x} = \vec{0}$) is also a linear dependence relations among the columns of U (the same $\vec{x}$ satisfies $U\vec{x} = \vec{0}$ since U and A are row equivalent).

For the same reason, the non-pivot columns of A are linear combinations of the pivot columns of A.

By discarding the non-pivot columns of A, we obtain a basis for $\mathrm{Col}(A)$. $\square$

Example. Find basis for the column spaces of

$$A = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 2 & 1 & 7 \\ 2 & 2 & 2 & 8 \end{bmatrix}.$$

The reduced row echelon form for A is

$$U = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Observe that the second column of U is 1 times the first column of U.

Is the second column of A the *same* scalar multiple of the first column of A? Yes it is.

The fourth column of U is 3 times the first column of U plus the third column of U.

Is the fourth column of A the *same* linear combination of the first and third columns of A? Yes, it is!

So the second and fourth columns of A are unnecessary as part of a spanning set.

A basis for the column space of A is the set

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\},$$

the first and third columns of A (NOT U), the pivot columns of A.