Math 313 Lecture #18
§4.4: Coordinate Systems

Now that we know how to find a basis of a vector space or its subspaces, we will learn why a basis is important.

We are familiar with the (x, y)-coordinates of the plane: the x and y quantities represent horizontal and vertical displacements from a starting point (we call the origin).

A basis of a vector space is the mechanism by which we impose coordinates on a vector space.

The following result is the key to this mechanism.

Theorem 7 (The Unique Representation Theorem). Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ be a basis for a vector space V. Then for each $\vec{x}$ in V there exists a unique set of weights $c_1, c_2, \dots, c_n$ such that

$$\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n.$$

Proof. Let $\vec{x}$ in V be arbitrary.

Since $\mathcal{B}$ spans V, there exists a set of weights $c_1, c_2, \dots, c_n$ such that

$$\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n.$$

To show the uniqueness of the set of weights, we suppose there is another set of weights $d_1, d_2, \dots, d_n$ such that

$$\vec{x} = d_1\vec{b}_1 + d_2\vec{b}_2 + \cdots + d_n\vec{b}_n.$$

Subtracting the one linear combination for $\vec{b}$ from the other gives

$$\vec{0} = (c_1 - d_1)\vec{b}_1 + (c_2 - d_2)\vec{b}_2 + \cdots + (c_n - d_n)\vec{b}_n.$$

Linear independence of $\mathcal{B}$ implies that the weights in this linear combination are all zero, so that $c_i = d_i$ for all $i = 1, 2, \dots, n$.

Thus there is a unique set of weights $c_1, c_2, \dots, c_n$ the representation of $\vec{x}$ in terms of the basis $\mathcal{B}$ is unique. □

Let us review a familiar example of basis and coordinates.

Example. For the standard basis $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ of $\mathbb{R}^n$, the unique representation of $\vec{x}$ in $\mathbb{R}^n$ is

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$$

where the unique weights $x_1, x_2, \dots, x_n$ are the *standard coordinates* of $\vec{x}$.

We can extract idea of standard coordinates and apply it to any basis of any vector space.

Definition. Suppose $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n\}$ is a basis for a vector space V. The **coordinates of $\vec{x}$ relative to the basis $\mathcal{B}$** (or the **$\mathcal{B}$-coordinates of $\vec{x}$**) are the unique weights $c_1, c_2, \dots, c_n$ for which

$$\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n.$$

Notationally we write and say that

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

is the **coordinate vector of** $\vec{x}$ (**relative to** $\mathcal{B}$), or the $\mathcal{B}$**-coordinate vector of** $\vec{x}$.

Example. The set

$$\mathcal{B} = \{\vec{b}_1, \vec{b}_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

form a basis for $\mathbb{R}^2$ because the matrix $A = [\vec{b}_1, \vec{b}_2]$ is invertible (which by the Inverse Matrix Theorem says that the columns span and are linearly independent).

What are the $\mathcal{B}$-coordinates of $\vec{e}_1$? We are asked to solve the vector equation

$$c_1\vec{b}_1 + c_2\vec{b}_2 = \vec{e}_1$$

or in other notation, the matrix equation $A\vec{c} = \vec{e}_1$.

This we do by row reduction of the augmented matrix to obtain $c_1 = 1/2$ and $c_2 = 1/2$, so that

$$[\vec{e}_1]_{\mathcal{B}} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}.$$

The basis $\mathcal{B}$ gives a different "view" of the plane, in which $\vec{b}_1$ and $\vec{b}_2$ correspond to the dials a different kind of Etch-A-Sketch (one dial moves northeast-southwest, the other dial moves southeast-northwest). / / / /

There is an important aspect of coordinates that is implicit in the above example that we extract.

In finding the coordinates of $\vec{x}$ in $\mathbb{R}^n$ relative to a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$, we solved the equation

$$c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n = \vec{x}.$$

If we set $P_{\mathcal{B}}$ to be the matrix $\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{bmatrix}$, then the above vector equation in matrix notation is

$$\vec{x} = P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}.$$

We call the matrix $P_{\mathcal{B}}$ the **change-of-coordinates** matrix form the basis $\mathcal{B}$ to the standard basis of $\mathbb{R}^n$.

The matrix $P_{\mathcal{B}}$ is invertible (why?) and gives the change of coordinates from the standard basis to the basis $\mathcal{B}$.

The Coordinate Mapping. The choice of a basis $\mathcal{B}$ for a vector space V gives a transformation $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ from the possible unfamiliar vector space V (the domain) to the familiar vector space $\mathbb{R}^n$ (the codomain).

What are the properties of this transformation? Is it linear? Is it one-to-one? Is it onto?

Theorem 8. For a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ of a vector space V, the coordinate mapping $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

Proof. To show that the coordinate mapping is linear, suppose there are two vectors in V, say

$$\vec{u} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n,$$
$$\vec{v} = d_1\vec{b}_1 + d_2\vec{b}_2 + \cdots + d_n\vec{b}_n.$$

Then

$$\vec{u} + \vec{v} = (c_1 + d_1)\vec{b}_1 + (c_2 + d_2)\vec{b}_2 + \cdots + (c_n + d_n)\vec{b}_n.$$

It follows that

$$[\vec{u} + \vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 + d_1 \\ c_2 + d_2 \\ \vdots \\ c_n + d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} = [\vec{u}]_{\mathcal{B}} + [\vec{v}]_{\mathcal{B}}.$$

For a scalar α we have $\alpha\vec{u} = \alpha c_1\vec{b}_1 + \alpha c_2\vec{b}_2 + \cdots + \alpha c_n\vec{b}_n$, and so

$$[\alpha\vec{u}]_{\mathcal{B}} = \begin{bmatrix} \alpha c_1 \\ \alpha c_2 \\ \vdots \\ \alpha c_n \end{bmatrix} = \alpha \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \alpha[\vec{u}]_{\mathcal{B}}.$$

Thus we have shown that the coordinate mapping is linear.

The proofs of the one-to-one and the onto are homework problems (#23,24). □

The one-to-one linear coordinate mapping $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ from a vector space V with basis $\mathcal{B}$ onto $\mathbb{R}^n$ is an example of a important type of transformation called an *isomorphism*.

It means that the possibly less familiar vector space V is just a "copy" of the more familiar $\mathbb{R}^n$: every vector space calculation in V (i.e., linear combinations) is accurately reproduced in $\mathbb{R}^n$.

Example. We have seen that the set of polynomials $\mathcal{B} = \{1, t, \ldots, t^n\}$ is a basis for the less familiar vector space $\mathbb{P}_n$.

The coordinate mapping $p(t) \mapsto [p(t)]_{\mathcal{B}}$ is the isomorphism

$$c_0 1 + c_1 t + \cdots + c_n t^n \mapsto \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

The image of the basis vector 1 is the vector $\vec{e}_1$, the image of the basis vector t is e_2, which continues, ending with the image of the basis vector t^n is e_{n+1}.

So $\mathbb{P}_n$ is nothing more than the vector space $\mathbb{R}^{n+1}$ in disguish!