Math 313 Lecture #19
§4.5: The Dimension of a Vector Space

Recall that the coordinate mapping is an isomorphism from a vector space with a basis of n vectors onto $\mathbb{R}^n$.

Could there be a different basis of V that had more or less than n vectors?

Theorem 9. If a vector space V has a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ of n vectors, then any set of vectors in V containing more than n vectors is linearly dependent.

Proof. Suppose $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ is a set of vectors in V with $p > n$.

The p coordinate vectors $[\vec{u}_1]_\mathcal{B}, [\vec{u}_2]_\mathcal{B}, \ldots, [\vec{u}_p]_\mathcal{B}$ are linearly dependent in $\mathbb{R}^n$ because $p > n$ (more columns than rows).

So there are weights $c_1, c_2, \ldots, c_p$, not all zero, such that

$$c_1[\vec{u}_1]_\mathcal{B} + c_2[\vec{u}_2]_\mathcal{B} + \cdots + c_p[\vec{u}_p]_\mathcal{B} = \vec{0}.$$

Since the coordinate mapping is linear we have that

$$[c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_p\vec{u}_p]_\mathcal{B} = \vec{0}.$$

Since the coordinate mapping is one-to-one we obtain that

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_p\vec{u}_p = \vec{0}.$$

With at least one of the weights $c_1, c_2, \ldots, c_p$ not zero, we conclude that $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_p\}$ is linearly dependent. □

An immediate consequence of Theorem 9 is that any basis of V cannot have more than n vectors in it.

Could a basis of V have less than n vectors?

Theorem 10. If a vector space V has a basis of n vectors, then every basis of V has n vectors.

Proof. Let $\mathcal{B}_1$ be a basis of n vectors, and let $\mathcal{B}_2$ be an arbitrary basis of V.

If $\mathcal{B}_2$ has more than n vectors, then by Theorem 9, the set $\mathcal{B}_2$ is linearly dependent and cannot be a basis.

So $\mathcal{B}_2$ has no more than n vectors in it.

Now suppose that the basis $\mathcal{B}_2$ has less than n vectors in it.

Again by Theorem 9, the set $\mathcal{B}_1$ would be linearly dependent and cannot be a basis.

So $\mathcal{B}_2$ has no less than n vectors in it.

Thus the basis $\mathcal{B}_2$ has exactly n vectors in it. □

Once we know that a nonzero vector space has a basis with a finite number of vectors in it, then every basis of that vector space has the same number of vectors in it.

We can show that a nonzero vector space has a basis with a finite number of vectors in it, by finding a finite spanning set S for the vector space (since by the Spanning Set Theorem, a subset of S is a basis).

We give the number of vectors common to all bases of a vector space a name.

Definitions. We say that a vector space V is spanned by a finite set is **finite-dimensional**, and the **dimension** of V, written as $\dim V$, is the number of vectors in any basis of V.

The dimension of the zero vector space $\{\vec{0}\}$ is $\dim V = 0$.

A vector space not spanned by a finite number of vectors in it, is said to be **infinite-dimensional**.

Examples. (a) The standard basis $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ of $\mathbb{R}^n$ contains n vectors, and so $\dim \mathbb{R}^n = n$.

(b) A basis for $\mathbb{P}_n$ is the set of $n+1$ vectors $\{1, t, \ldots, t^n\}$, and so $\dim \mathbb{P}_n = n+1$.

(c) What is the dimension of $\mathbb{R}^{2\times 2}$ (or what the text denotes $M_{2\times 2}$)?

For an arbitrary 2×2 matrix A we have

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

We expressed every 2×2 matrix as a linear combination of 4 matrices (the span of four matrices).

Are the four matrices linearly independent?

Yes, they are, because setting to the zero matrix the linear combination of the four matrices with weights a, b, c, d shows that $a = b = c = d = 0$.

Thus the four matrices are a basis $\mathcal{B}$ for $\mathbb{R}^{2\times 2}$, and so $\dim, \mathbb{R}^{2\times 2} = 4$.

What is the coordinate mapping from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^4$ for this basis? It is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto [A]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

The vector space $\mathbb{R}^{2\times 2}$ is just $\mathbb{R}^4$ in disguise.

(d) Is the vector space $\mathbb{P}$ of all polynomials finite dimensional?

It is not spanned by the linearly independent set $\{1, t, \ldots, t^n\}$ for each choice of n because t^{n+1} is not in its span.

Thus $\mathbb{P}$ is infinite dimensional.

(e) Is the vector space $C[a, b]$ finite dimensional?

It is not spanned by the linearly independent set $\{1, t, \ldots, t^n\}$ (with their domains restricted to $[a, b]$) for each n because t^{n+1} (with domain restricted to $[a, b]$) is not in its span.

Thus $C[a, b]$ is infinite dimensional. / / / /

What is the relationship between the dimension of a vector space V and the dimension of a subspace H of V?

Theorem 11. *Let H be a subspace of a finite dimensional vector space V. Any linearly independent set in H can be expanded, if needed, to a basis of H. Also, H is finite dimensional and $\dim H \leq \dim V$.*

Proof. If $H = \{\vec{0}\}$, then $\dim H = 0 \leq \dim V$.

So suppose $H \neq \{\vec{0}\}$, and let $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ be a linearly independent set in H.

If S spans H, then S is a basis for H.

If S does not span H, then there is a vector $\vec{u}_{k+1}$ in H but not in the span of S.

The expansion of S obtained by adding $\vec{u}_{k+1}$ to it results in the new set $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k, \vec{u}_{k+1}\}$, which is linearly independent because no vector in it is a linear combination of the vectors that precede it (Theorem 4).

If the new set does not span H, we continue to add vectors until we obtain a linearly independent set that spans H.

The number of vectors in the expansion of S cannot exceed the dimension of V, because any set of vectors with more than $\dim V$ vectors in it is linearly dependent (by Theorem 9).

We therefore obtain that $\dim H \leq \dim V$. □

When we know the dimension p of a finite dimensional vector space (or a nonzero subspace of it), the search for a basis for the vector space need only check one of the linear independence of p vectors or the span of p vectors (but not both) to get a basis.

Theorem 12 (The Basis Theorem)**.** *Let V be a p-dimensional vector space with $p \geq 1$. Any linearly independent set of p vectors in V is a basis for V, and any set of p vectors that span V is a basis for V.*

Proof. If a linearly independent set of p vectors did not span V, then applying Theorem 10 with $H = V$, we can enlarge the linearly independent set of p vectors to a linearly independent set of $p + 1$ vectors in V.

But V is p-dimensional, and any set of $p+1$ vectors in it is linearly dependent by Theorem 9.

This contradiction implies that the set of p linearly independent vectors does span V, and is hence a basis.

On the other hand, suppose a set S of p vectors spans V.

By the Spanning Set Theorem, a subset S' of S is a basis for V.

The number of vectors in the basis S' is p by Theorem 10 because $\dim V = p$.

The only subset of S' with p elements is S, and so $S' = S$, meaning that S is a basis for V. $\square$