Math 313 Lecture #20
§4.6: Rank

We have seen the fundamental role that pivot positions play in solving $A\vec{x} = \vec{b}$.

With the framework of vector spaces, we will find several useful relationships among the rows and columns of a matrix, some of which we have seen before.

The Row Space. An $m \times n$ matrix has m rows where each row consists of n entries.

Thus the rows of A belong to the vector space $\mathbb{R}^n$.

The subspace of $\mathbb{R}^n$ spanned by the row of A is the called the **row space** of A, and is denoted $\mathrm{Row}\, A$.

Notice that the row space of A is the same as the column space of A^T.

We can find a basis for the row space of A by row reduction of A.

Theorem 13. Two row equivalent matrices have the same row space. If B is an echelon form for A, then the nonzero rows of B are a basis for the row space of A.

Proof. The first statement follows from recognizing that row operations preserve the row space.

If B is row equivalent to A, then the rows of B are linear combinations of the rows of A.

This means that the row space of B is a subset of the row space of A.

Since A is row equivalent to B, the same argument shows that the row space of A is a subset of the row space of B.

Thus A and B have the same row space.

The second statement follows from the echelon form B of A: the nonzero rows of B are linearly independent since each nonzero row is not a linear combination of the rows above it.

Since the nonzero rows of B span the row space of B, the nonzero rows of B are a basis for the row space of B, and hence for the row space of A. $\square$

Example. An echelon form for the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 & 6 & 1 \\ 2 & 6 & 1 & 11 & 1 \\ 3 & 9 & 2 & 17 & 2 \\ 6 & 18 & 2 & 33 & 2 \end{bmatrix}$$

is

$$B = \begin{bmatrix} 1 & 3 & 1 & 6 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for $\mathrm{Row}(A)$ is the set of the three nonzero rows of B, and so $\dim \mathrm{Row}(A) = 3$.

Could the first three rows of A be a basis for $\text{Row}(A)$?

No, because row 3 of A is the sum of rows 1 and 2 of A, meaning that the first three rows of A form a linearly dependent set.

What is a basis for $\text{Col}(A)$?

Since the pivot positions of B occur in the first, third, and fourth columns, a basis for $\text{Col}(A)$ is the set of the first, third, and fourth columns of A, and so $\dim \text{Col}(A) = 3$.

Is it just coincidence that the row space and column space of A have the same dimension? Or it is a theorem?

It is a theorem (given below) because both dimensions are determined by the number of pivot positions.

What is a basis for the null space of A?

To better answer this question we find the reduced row echelon form of A, which is

$$U = \begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The basic variables are x_1, x_3, and x_4, while the free variables are $x_2 = s$ and $x_5 = t$.

The null space of A is given by

$$\vec{x} = \begin{bmatrix} -3s \\ s \\ -t \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

A basis for $\text{Nul}(A)$ is the set of two vectors given in the linear combination, and so $\dim \text{Nul}(A) = 2$.

Is it just coincidence that the sum of the dimensions of the null space and the column space equals the number of columns? Or it is a theorem?

It is a theorem (which we will present in a moment, after defining an important quantity associated with a matrix).

Definition. The **rank** of an $m \times n$ matrix is the dimension of the column space of A, and is denoted by $\text{rank}(A)$.

Since the row space of A^T is the same as the column space of A, the rank of A is the dimension of the row space of A^T.

Also, the rank of A^T is the same as the dimension of the row space of A.

The dimension of the null space of A is sometimes called the **nullity** of A.

Theorem 14 (The Rank Theorem). For any $m \times n$ matrix A, the dimensions of Row(A) and Col(A) are the same, and there holds

$$\operatorname{rank}(A) + \dim N(A) = n.$$

Proof. We have already seen the reason why the dimensions of Row(A) and Col(A) are the same: both are determined by the number of pivots of A.

The dimension of Col(A) is the number of pivots of A, or the number of basic variables.

On the other hand, the dimension of Nul(A) is the number of free variables.

The sum of the number of basic variables and free variables is the number of variables, which is the same as the number of columns of A. □

The Rank Theorem is useful for non-square matrices.

It can place some restrictions on the rank and nullity of a matrix by only knowing the size of the matrix.

Examples. (a) For a 8×6 matrix A we have that

$$\operatorname{rank}(A) + \dim \operatorname{Nul}(A) = 6.$$

How big can the rank of A be? It can be no bigger than 6. If it is 6, then the nullity of A is 0.

How small can the rank of A be? It can be 0, which means that $A = 0$, and so the nullity of A is 6.

How big can the nullity of A be? It can be no bigger than 6. If it is 6, then the rank of A is 0, which means A is the 0 matrix.

How small can the nullity of A be? It can be 0, which means that the rank of A is 6.

(b) For a 4×7 matrix B, we have that

$$\operatorname{rank}(B) + \dim \operatorname{Nul}(B) = 7.$$

How big can the rank of B be? It can not be as big as 7, why? No enough pivots. The rank of B can be as big as 4, the largest number of pivots possible.

How small can the rank of B be? It can be 0, in which case $A = 0$ and the nullity of A is 4.

How big can the nullity of B be? It can be as big as the largest number of free variables possible, which is 7. In this case $A = 0$ and the rank of B is 0.

How small can the nullity of B be? It cannot be 0, because not every column can be a pivot column. The smallest possible nullity is the smallest number of free variables possible, which is $7 - 4 = 3$. ////

For square matrices, the notions of subspace, basis, and rank add more equivalent statements to the Inverse Matrix Theorem.

The Inverse Matrix Theorem (Continued). The invertibility of an $n \times n$ matrix is equivalent to any of the following statements.

m. The columns of A form a basis of $\mathbb{R}^n$.
n. $\mathrm{Col}(A) = \mathbb{R}^n$.
o. $\dim \mathrm{Col}(A) = n$.
p. $\mathrm{rank}(A) = n$.
q. $\mathrm{Nul}(A) = \{\vec{0}\}$.
r. $\dim \mathrm{Nul}(A) = 0$.