Math 313 Lecture #21
§5.1: Eigenvectors and Eigenvalues

We have learned many things about solving the matrix equation $A\vec{x} = \vec{b}$.

Now we turn our attention to solving another problem in linear algebra, namely the equation

$$A\vec{x} = \lambda\vec{x}$$

for an $n \times n$ matrix A and a scalar λ.

Example. For the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

and the scalar $\lambda = 3$, the vector

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

satisfies $A\vec{x} = \lambda\vec{x}$ because

$$A\vec{x} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \text{ and } \lambda\vec{x} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

Are there other choices of $\vec{x}$ and λ that work too?

Certainly the choice of $\vec{x} = \vec{0}$ works for any scalar λ, but this does not depend on what A is, so we will exclude this *trivial* case.

Definitions. An **eigenvector** of an $n \times n$ matrix A is a **NONZERO** vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ.

An **eigenvalue** of an $n \times n$ matrix A is a scalar (could be zero) such that there is a **NONZERO** vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$; such a vector is called an eigenvector corresponding to λ.

What is the significance of the eigenvectors and eigenvalues of A?

Example (Continued). We know that $\lambda = 3$ is an eigenvalue of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix},$$

and a corresponding eigenvector is

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The eigenvector satisfies the equation $A\vec{x} = 3\vec{x}$, or rewritten,

$$A\vec{x} - 3\vec{x} = \vec{0}.$$

We would like to factor out the common $\vec{x}$, but we might mistakenly write

$$(A - 3)\vec{x} = \vec{0}.$$

What is incorrect with this equation? There is no way to subtract the scalar 3 from the 2×2 matrix A.

Instead, we write $3\vec{x} = 3I\vec{x}$ for the 2×2 identity matrix I, so that

$$(A - 3I)\vec{x} = \vec{0}.$$

This homogeneous equation has a nontrivial solution in $\vec{x}$.

By the Inverse Matrix Theorem the matrix $A - 3I$ is not invertible, which we verify directly: the determinant of

$$A - 3I = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$$

is 0, and so $A - 3I$ is not invertible.

The nonzero vector $\vec{x}$ belongs to the null space of $A - 3I$; in particular, $\vec{x}$ is a basis for $\text{Nul}(A - 3I)$ because

$$A - 3I \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

so that u_1 is a basic variable, $u_2 = t$ is a free variable, and the solution set of $(A-3I)\vec{u} = \vec{0}$ consists of the vectors

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix} = t\vec{x}.$$

Are there other values of λ for which $A - \lambda I$ is not invertible? Yes, $\lambda = -1$.

An eigenvector of A corresponding to $\lambda = -1$ is a nontrivial solution of $(A + I)\vec{u} = \vec{0}$, namely

$$\vec{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

It so happens that this $\vec{x}$ is a basis for $\text{Nul}(A + I)$. / / / /

We see in this example that eigenvectors of A are basis vectors of null spaces of matrices of the form $A - \lambda I$.

Thus λ is an eigenvalue of A if and only if the homogeneous equation $(A - \lambda I)\vec{x} = \vec{0}$ has a nontrivial solution.

For an eigenvalue λ of A we call the null space of $A - \lambda I$ the **eigenspace** of A corresponding to λ.

For a real eigenvalue λ of a matrix A, the matrix transformation $\vec{x} \mapsto A\vec{x} = \lambda\vec{x}$ acts like dilation/contraction on the eigenspace $\text{Nul}(A - \lambda I)$.

At this point, we only have the method of guess and check to find eigenvalues of a square matrix.

But for a certain type of square matrix, the eigenvalues are readily found.

Theorem 1. The eigenvalues of a triangular square matrix are its diagonal entries.

WARNING: this does not say that we can find eigenvalues by row reduction. A counterexample to this is the invertible matrix

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

whose eigenvalues are 3 and -1, but the eigenvalues of the I (the reduced row echelon form of A) are 1 and 1.

Proof. Suppose A is in triangular form with diagonal entries a_{ii} for $i = 1, 2, \ldots, n$.

For a fixed i set $\lambda = a_{ii}$.

One of the diagonal entries of $A - \lambda I$ is zero (namely the (i, i) entry), and so $\det(A - \lambda I) = 0$.

By the Inverse Matrix Theorem, there is a nontrivial solution $\vec{x}$ of $(A - \lambda I)\vec{x} = \vec{0}$.

This means that $A\vec{x} = \lambda\vec{x}$, and so λ is an eigenvalue. □

The eigenvectors of a square matrix associated to distinct eigenvalues enjoy a valuable property.

Theorem 2. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r$ are eigenvectors corresponding to distinct eigenvalues of a square matrix, then the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$ is linearly independent.

Proof. We prove this by way of contradiction, by supposing that the set of eigenvectors is linearly dependent.

Since $\vec{v}_1 \neq \vec{0}$, we can apply the linear dependent Theorem: there is a least index p such that the eigenvector v_{p+1} is a linear combination of the preceding p eigenvectors.

The choice of the least index p means that the set of eigenvectors $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is linearly independent (for otherwise there would be a smaller choice of p).

There are weights $c_1, c_2, \ldots, c_p$ such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{v}_{p+1}.$$

To this we apply the matrix transformation $\vec{x} \mapsto A\vec{x}$ to obtain

$$c_1A\vec{v}_1 + c_2A\vec{v}_2 + \cdots + c_pA\vec{v}_p = A\vec{v}_{p+1}.$$

Since $A\vec{v}_i = \lambda_i\vec{v}_i$ for each $i = 1, 2, \ldots, p + 1$, we obtain

$$c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + \cdots + c_p\lambda_p\vec{v}_p = \lambda_{p+1}\vec{v}_{p+1}.$$

Multiplying the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{v}_{p+1}$ through by λ_{p+1} gives

$$c_1\lambda_{p+1}\vec{v}_1 + c_2\lambda_{p+1}\vec{v}_2 + \cdots + c_p\lambda_{p+1}\vec{v}_p = \lambda_{p+1}\vec{v}_{p+1}.$$

Subtracting this equation from $c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + \cdots + c_p\lambda_p\vec{v}_p = \lambda_{p+1}\vec{v}_{p+1}$ gives

$$c_1(\lambda_1 - \lambda_{p+1})\vec{v}_1 + c_2(\lambda_2 - \lambda_{p+1})\vec{v}_2 + \cdots + c_p(\lambda_p - \lambda_{p+1})\vec{v}_p = \vec{0}.$$

The linear independence of the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ now implies that $c_i(\lambda_i - \lambda_{p+1}) = 0$ for each $i = 1, 2, \ldots, p$.

Thus $c_i = 0$ for each $i = 1, 2, \ldots, p$ since the eigenvalues are distinct.

This implies that $v_{p+1} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0}$.

[What is the contradiction here?]

But $\vec{v}_{p+1}$ is an eigenvector, meaning it is nonzero, a contradiction. $\square$