Math 313 Lecture #22
§5.2: The Characteristic Equation

The only method we have so far for finding the eigenvalue of a square matrix is guess and check.

We will develop a sure shot method today, which we introduce by way of example.

Example. Last time we learned that the eigenvalues of

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

are $\lambda = 3$ and $\lambda = -1$.

For either value of λ, we know that the matrix $A - \lambda I$ is not invertible (so that the homogeneous equation $(A - \lambda I)\vec{x} = \vec{0}$ has a nontrivial solution – an eigenvector corresponding to λ).

By the Inverse Matrix Theorem, the noninvertibility of $A - \lambda I$ is the same as

$$\det(A - \lambda I) = 0.$$

Computing the determinant of

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix}$$

by the formula for a 2×2 matrix gives a *quadratic polynomial* in λ,

$$\det(A - \lambda I) = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3.$$

This polynomial factors as $(\lambda - 3)(\lambda + 1)$, so that the eigenvalues of A are the roots of the quadratic polynomial, namely $\lambda = 3$ and $\lambda = -1$.

How can we check our eigenvalues? We compute the determinant of $A - \lambda I$ for each value of λ: the determinants of

$$A - 3I = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \text{ and } A + I = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

are both zero.

We can then proceed to find eigenvectors by finding a basis for each eigenspace. / / / /

The Characterization Equation. From the motivating example, we can now develop the sure shot method for finding the eigenvalues of a square matrix.

The polynomial equation $\det(A - \lambda I) = 0$ is called the **characteristic equation** of A, while the polynomial $\det(A - \lambda I)$ is called the **characteristic polynomial** of A.

A scalar λ is an eigenvalue of A if and only if λ satisfies the characteristic equation $\det(A - \lambda I) = 0$.

Finding eigenvalues of A it simply a matter of finding the roots of the characteristic polynomial of A.

Example. Find the eigenvalues of

$$A = \begin{bmatrix} 2 & 0 & -2 \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{bmatrix}.$$

We form the matrix

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 0 & -2 \\ 1 & 1-\lambda & -2 \\ 1 & 0 & -1-\lambda \end{bmatrix},$$

and compute its determinant by cofactor expansion, in this case, down the second column (by cofactor expansion only, because row reduction gets messy with the λ's in the matrix):

$$\begin{aligned} \det(A - \lambda I) &= (1-\lambda)\begin{vmatrix} 2-\lambda & -2 \\ 1 & -1-\lambda \end{vmatrix} \\ &= (1-\lambda)\big[(2-\lambda)(-1-\lambda)+2\big] \\ &= (1-\lambda)\big[-2-2\lambda+\lambda+\lambda^2+2\big] \\ &= (1-\lambda)\big[\lambda^2-\lambda\big] \\ &= (1-\lambda)\lambda(\lambda-1) \\ &= -\lambda(\lambda-1)^2. \end{aligned}$$

The eigenvalues of A are the roots of the cubic $-\lambda(\lambda-1)^2$, namely $\lambda = 0$ and $\lambda = 1$.

What does the 0 eigenvalue of A say about A? That $A = A - 0I$ is noninvertible.

The converse holds too: If A is noninvertible, then $A - 0I$ is noninvertible.

This adds yet another statement to the Inverse Matrix Theorem (it is statement s.).

What do we make of the factor $(\lambda-1)^2$ of the characteristic polynomial? That the eigenvalue $\lambda = 1$ occurs twice.

We say that the **algebraic multiplicity** of the eigenvalue $\lambda = 1$ is two, while the algebraic multiplicity of the eigenvalue $\lambda = 0$ is one (because the factor λ occurs only once in the characteristic polynomial).

Similarity. What can we say about two matrices that have the same eigenvalues?

Example. The matrices

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix},\ B = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$$

have the same characteristic polynomial $\lambda^2 - 2\lambda - 3$, and so have the same eigenvalues.

How might A and B be related?

The invertible matrix

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

satisfies the equation $B = P^{-1}AP$ which we can verify by computing PB and AP:

$$PB = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix},$$

$$AP = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix}.$$

Example. The matrices

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \ B = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

also have the same characteristic polynomial $(3 - \lambda)^2$, and so have the same eigenvalues.

But there is no invertible matrix P such that $B = P^{-1}AP$.

Why? Because for

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

we have that

$$PB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3a & a+3b \\ 3c & c+3d \end{bmatrix},$$

$$AP = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix},$$

which forces $a = 0$ and $c = 0$, and hence P is not invertible.

Definitions. For two $n \times n$ matrices A and B, we say that A is **similar** to B if there exists an invertible $n \times n$ matrix P such that $B = P^{-1}AP$.

Writing $Q = P^{-1}$ we have $A = Q^{-1}BQ$, and so B is similar to A.

We can thus simple say that A and B are **similar**.

For a fixed invertible matrix P, the mapping taking A in $M_{n\times n}$ to $P^{-1}AP$ in $M_{n\times n}$ is called a **similarity transformation**.

Theorem 4. If two $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial (and hence the same eigenvalues with the same algebraic multiplicities).

Proof. Suppose A and B are similar.

Then there exists an invertible matrix P such that $B = P^{-1}AP$.

The

$$B - \lambda I = P^{-1}AP - \lambda I = P^{-1}AP - \lambda P^{-1}IP = P^{-1}\big(A - \lambda I\big)P.$$

Computing the characteristic polynomial of B we have

$$\det(B - \lambda I) = \det\big(P^{-1}(A - \lambda I)P\big) = \det(P^{-1})\det(A - \lambda I)\det(P).$$

Since $\det(P^{-1})\det(P) = \det(P^{-1}P) = \det(I) = 1$, we arrive at $\det(B-\lambda I) = \det(A-\lambda I)$.

Therefore A and B have the same characteristic polynomial. $\square$