Math 313 Lecture #23
§5.3: Diagonalization

A Motivational Example. Recall that the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \text{ are } \lambda_1 = 3,\ \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\quad \lambda_2 = -1,\ \vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Let P be the matrix whose columns are the eigenvectors of A:

$$P = [\vec{v}_1\ \vec{v}_2] = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}.$$

The matrix P is invertible since its columns are linearly independent; its inverse is

$$P^{-1} = \frac{1}{-4}\begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{bmatrix}.$$

The matrix $P^{-1}AP$ is similar to the matrix A.

What is $P^{-1}AP$? Well,

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 3/2 & 3/4 \\ -1/2 & 1/4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 3/2+3/2 & 3/2-3/2 \\ -1/2+1/2 & -1/2-1/2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}. \end{aligned}$$

Recognize the entries on the diagonal of this matrix? They are the eigenvalues of A in the order in which their eigenvectors were placed in P!

Definitions. An $n \times n$ matrix A is **diagonalizable** if it is similar to a diagonal matrix.

We call an invertible matrix P for which $P^{-1}AP$ is diagonal, a **diagonalizing matrix** for A.

Is every square matrix diagonalizable?

Theorem 5. An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

Proof. For any invertible matrix P with columns $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ and any diagonal matrix D with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$, we have

$$AP = A\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} = \begin{bmatrix} A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_n \end{bmatrix},$$

$$PD = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \end{bmatrix}.$$

Suppose that A has n linearly independent eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.

Let λ_i be the eigenvalue of A corresponding to $\vec{v}_i$, i.e., $A\vec{v}_i = \lambda_i \vec{v}_i$.

Then $AP = PD$.

Why is P invertible? Because its columns form a linearly independent set, so by the Inverse Matrix Theorem, P is invertible.

Thus we have $D = P^{-1}AP$, and so A is diagonalizable with diagonalizing matrix P.

Now suppose that A is diagonalizable.

Then there is an invertible matrix P with columns $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ and a diagonal matrix D with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $D = P^{-1}AP$.

So $PD = AP$, which means $A\vec{v}_i = \lambda_i \vec{v}_i$ for each $i = 1, 2, \ldots, n$, that is, each $\vec{v}_i$ is an eigenvector of A.

Since P is invertible, the columns of P form an independent set of vectors, and therefore A has n linearly independent eigenvectors. □

Theorem 6. If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

Proof. Any set of n eigenvectors corresponding to the n distinct eigenvalues are linearly independent, and so A is diagonalizable by Theorem 5. □

Example. Is $A = \begin{bmatrix} -1 & -3 & -4 \\ 1 & 3 & 2 \\ 1 & 1 & 3 \end{bmatrix}$ diagonalizable?

The characteristic polynomial of A is

$$p(\lambda) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = -(\lambda - 1)(\lambda - 2)^2.$$

So the eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 2$.

Row reduction of $A - I$ gives the eigenspace of A belonging the eigenvalue 1 of algebraic multiplicity 1:

$$A - I = \begin{bmatrix} -2 & -3 & -4 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Nul}(A - I) = \text{Span}\left(\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right).$$

The **geometric multiplicity** of the eigenvalue 1 is 1, the dimension of $\text{Nul}(A - I)$.

An eigenvector of A belonging to $\lambda_1 = 1$ is $\vec{v}_1 = [-2\ 0\ 1]^T$.

Row reduction of $A - 2I$ gives the eigenspace of A belonging to eigenvalue 2 of algebraic multiplicity 2:

$$A - 2I = \begin{bmatrix} -3 & -3 & -4 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Nul}(A - 2I) = \text{Span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right).$$

The **geometric multiplicity** of the eigenvalue 2 is not 2 but is 1, the dimension of $\text{Nul}(A - 2I)$.

An eigenvector of A belonging to the eigenvalue 2 is $\vec{v}_2 = [-1\ 1\ 0]^T$.

The two eigenvectors $\vec{v}_1, \vec{v}_2$ are linearly independent.

Is there a third eigenvector $\vec{v}_3$ for which the set of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is linearly independent?

If there were, then $A\vec{v}_3 = \lambda\vec{v}_3$ for an eigenvalue λ of A, which would mean that $\vec{v}_3 \in \text{Nul}(A - I)$ or $\vec{v}_3 \in \text{Nul}(A - 2I)$, hence $\vec{v}_3$ would be a nonzero scalar multiple of $\vec{v}_1$ or $\vec{v}_2$.

But then $\vec{v}_1, \vec{v}_2, \vec{v}_3$ would form a linearly dependent set.

So, A has only 2 linearly independent eigenvectors, and is not diagaonalizable.

Could an $n \times n$ matrix be diagonalizable when it does not have n distinct eigenvalues?

Theorem 7. Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

a. For each $k = 1, \ldots, p$, the geometric multiplicity of λ_k is less than or equal to its algebraic multiplicity.

b. The $n \times n$ matrix A is diagonalizable if and only if the sum of the geometric multiplicities of its eigenvalues equals n which happens if and only if the geometric multiplicity of each eigenvalue is equal to its algebraic multiplicity.

c. If A is diagonalizable and $\mathcal{B}_k$ is a basis of the eigenspace $\text{Nul}(A - \lambda_k I)$ for each k, then the union of the $\mathcal{B}_k$ is an eigenvector basis for $\mathbb{R}^n$.

Example. Is $A = \begin{bmatrix} 2 & -2 & 2 \\ 0 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix}$ diagonalizable?

The characteristic polynomial of A is $\det(A - \lambda I) = -(\lambda - 1)(\lambda - 2)^2$.

The eigenvalue $\lambda = 1$ has algebraic multiplicity 1.

Its geometric multiplicity of 1. Why? Because there is a linearly independent solution of $(A - I)\vec{x} = \vec{0}$, but no more than one.

The eigenvalue $\lambda = 2$ has algebraic multiplicity 2.

What is its geometric multiplicity?

We row reduce $A - 2I$ to find out:

$$A - 2I = \begin{bmatrix} 0 & -2 & 2 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

There are two free variables, and so $\dim \text{Nul}(A - 2I) = 2$, meaning the geometric multiplicity of $\lambda = 2$ is 2.

Thus the matrix A is diagonalizable.