Math 313 Lecture #24
§5.5: Complex Eignenvalues

Recall that the eigenvalues of an $n \times n$ matrix A with real entries are the roots of a polynomial of degree n and real coefficients.

It is therefore possible that some or all of the eigenvalues can be complex numbers.

To gain an understanding of what a complex valued eigenvalue means, we extend the domain and codomain of $\vec{x} \mapsto A\vec{x}$ from $\mathbb{R}^n$ to $\mathbb{C}^n$.

We do this because when λ is a complex valued eigenvalue of A, a nontrivial solution of $A\vec{x} = \lambda\vec{x}$ will be a complex valued vector in $\mathbb{C}^n$. [We will demonstrate this in a moment.]

We call λ a **complex eigenvalue** and $\vec{x}$ a **complex eigenvector**.

Example. Recall that the rotation matrix for a given angle θ is

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Does this rotation have a real eigenvalue/eigenvector pair? Only when θ is a multiple of π, because $R\vec{x}$ is a rotation of $\vec{x}$ by θ in the counterclockwise direction, and only when θ is multiple of π is $A\vec{x}$ a real scalar multiple of $\vec{x}$.

If we take $\theta = \pi/2$, then we have the rotation by $\pi/2$:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of A is $\lambda^2 + 1$ which has pure imaginary roots $\pm\sqrt{-1} = \pm i$.

An eigenvector for the eigenvalue i is a nonzero vector in the null space of

$$A - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}.$$

Does this matrix have determinant zero? Yes it does: $(-i)(-i) + 1 = i^2 + 1 = -1 + 1 = 0$.

That means that the two rows are linearly dependent. Is one row a scalar multiple of the other? real or complex scalar?

The second row of $A - iI$ is i times the first row, so we can eliminate the second row.

A complex eigenvector of A for the eigenvalue $\lambda = i$ is

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

An eigenvector for the eigenvalue $\lambda = -i$ is a nonzero vector in the null space of

$$A + iI = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}.$$

Does $A + iI$ have determinant zero? Yes, it does because $i^2 + 1 = -1 + 1 = 0$.

Thus one row of $A + iI$ is a scalar multiple of the other row: multiplying the first row by $-i$ gives the second row.

A complex eigenvector of A for the eigenvalue $\lambda = -i$ is

$$\vec{v}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

What do you notice about the eigenvalue/eigenvector pairs here? That the eigenvalues are complex conjugates of each other AND their eigenvectors are also complex conjugates of each other (where we understand the complex conjugate of a vector to be the complex conjugate of the entries). More about this in a minute.

The complex matrix

$$P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$$

is invertible because $ad - bc = i + i = 2i \neq 0$.

The invertible matrix P is a diagonalizing matrix for A:

$$P^{-1}AP = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Although the diagonal matrix is the "simplest view" of what the matrix A does as a linear transformation on $\mathbb{C}^2$, it hides what it does on $\mathbb{R}^2$. / / / /

There is a way to keep things in $\mathbb{R}^2$ instead of $\mathbb{C}^2$.

This way requires that we consider the real and imaginary parts of a complex eigenvector.

Every complex eigenvector $\vec{x}$ can be "pulled apart" into a real part and an imaginary part:

$$\vec{x} = \vec{a} + i\vec{b} = \mathrm{Re}\,\vec{x} + i\mathrm{Im}\,\vec{x}.$$

Example. Pulling apart a complex vector into its real and imaginary parts is

$$\vec{x} = \begin{bmatrix} 1+i \\ 2 \\ -i \\ 4-3i \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \\ -1 \\ -3 \end{bmatrix}.$$

So

$$\mathrm{Re}\,\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \ \mathrm{Im}\,\vec{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -3 \end{bmatrix}.$$

The complex conjugate of a complex number $z = a + bi$ is the complex number $\overline{z} = a - bi$.

This readily extends to vectors and matrices entry wise.

So, if $\lambda = a + bi$ is a complex eigenvalue of A and $\vec{x}$ is an eigenvector of A corresponding to λ, then

$$A\overline{\vec{x}} = \overline{A\vec{x}} = \overline{\lambda\vec{x}} = \overline{\lambda}\,\overline{\vec{x}}.$$

What is this saying? That $\overline{\lambda}$ is an eigenvalue of A and $\overline{\vec{x}}$ is an eigenvector of A corresponding to $\overline{\lambda}$.

Remembering seeing this in the previous example that the complex conjugate eigenvalues have complex conjugate eigenvectors?

When A is a real matrix, its complex eigenvalues occur in complex conjugate pairs, and the corresponding complex eigenvectors can be chosen to occur in complex conjugate pairs.

We identify a prototype of a real 2×2 matrix A whose eigenvalues are a complex conjugate pair.

Example. The characteristic polynomial of the matrix

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

with real a and b, not both zero, is

$$\det(C - \lambda I) = \begin{vmatrix} a-\lambda & -b \\ b & a-\lambda \end{vmatrix} = (a-\lambda)^2 + b^2 = \lambda^2 - 2a\lambda + a^2 + b^2.$$

By the quadratic formula, the roots of the characteristic polynomial are

$$\lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2+b^2)}}{2} = a \pm \sqrt{-b^2} = a \pm bi.$$

If we set $r = |\lambda| = \sqrt{a^2+b^2} \neq 0$ (the distance from the origin to the point (a,b) in the plane), and θ the angle between the positive horizontal axis and the ray from the origin to the point (a,b) (think polar coordinates!), then

$$C = r\begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

The matrix C as a linear transformation is the composition of a rotation by angle θ with a scaling by λ. ////

Every real 2×2 matrix A with complex conjugate eigenvalues is similar to such a C, with the columns of the invertible matrix P being the real and imaginary parts of a complex eigenvector.

Theorem 9. Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ with $b \neq 0$, and associated eigenvector $\vec{v}$. Then

$$A = PCP^{-1} \text{ for } C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

where $P = \begin{bmatrix} \operatorname{Re}\vec{v} & \operatorname{Im}\vec{v} \end{bmatrix}$.

Example. For

$$A = \begin{bmatrix} -7 & 15 \\ -6 & 11 \end{bmatrix},$$

find the matrix C for which $A = PCP^{-1}$.

The characteristic polynomial of A is

$$\det(A - \lambda I) = \lambda^2 - 4\lambda + 13.$$

The eigenvalues of A are

$$\lambda = \frac{4 \pm \sqrt{16 - 52}}{2} = 2 \pm 3i.$$

By Theorem 9, the matrix C for which $A = PCP^{-1}$ is

$$C = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}.$$

To verify this, we find P for which $A = PCP^{-1}$.

An eigenvector for $\lambda = 2 - 3i$ is a nonzero vector in the null space of

$$A - \lambda I = \begin{bmatrix} -9 + 3i & 15 \\ -6 & 9 + 3i \end{bmatrix}.$$

The matrix is noninvertible because its determinant is $-81 - 9 + 90 = 0$.

Elimination of the first row gives as an eigenvector

$$\vec{v} = \begin{bmatrix} 3 + i \\ 2 \end{bmatrix}.$$

We split this eigenvector into its real and imaginary parts that form the columns of

$$P = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}.$$

We then verify that $A = PCP^{-1}$ or the easier $AP = PC$:

$$AP = \begin{bmatrix} -7 & 15 \\ -6 & 11 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 5 & -1 \end{bmatrix},$$

$$PC = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 5 & -1 \end{bmatrix}.$$