Math 313 Lecture #25
§6.1: Inner Product, Length, and Orthogonality

We will now impose on the vector space $\mathbb{R}^n$ a structure that enables us to define geometric notions of length and angle between vectors.

We define the **inner product** of two vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ to be the scalar quantity

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = [u_1 \ u_2 \ \ldots \ u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots u_n v_n = \sum_{i=1}^{n} u_i v_i.$$

This inner product is also known as the **dot product** in $\mathbb{R}^n$.

Theorem 1. The inner product $\vec{u} \cdot \vec{v}$ on $\mathbb{R}^n$ satisfies

a. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$,
b. $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$,
c. $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$, and
d. $\vec{u} \cdot \vec{u} \geq 0$ and $\vec{u} \cdot \vec{u} = 0$ if and only if $\vec{u} = 0$.

The **length**, or **norm** of a vector $\vec{x}$ in $\mathbb{R}^n$ is the nonnegative quantity

$$\|\vec{u}\,\| = \sqrt{(\vec{u} \cdot \vec{u})} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}.$$

An important variation of this is the equation

$$\|\vec{u}\|^2 = \vec{u} \cdot \vec{u}.$$

For any scalar c, the length of $c\vec{u}$ is

$$\|c\vec{u}\| = \sqrt{(cu_1)^2 + (cu_2)^2 + \cdots + (cu_n)^2} = |c| \ \|\vec{u}\|.$$

A vector whose length is one is called a **unit vector**.

We always get a unit vector from a nonzero vector when we multiply the vector by the reciprocal of its length:

$$\left\| \frac{1}{\|\vec{u}\|} \vec{u} \right\| = \frac{1}{\|\vec{u}\|} \|\vec{u}\| = 1.$$

The **distance** between two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ is the quantity

$$\|\vec{u} - \vec{v}\,\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}.$$

Two vectors $\vec{u}$ and $\vec{v}$ form an angle $0 \leq \theta \leq \pi$. How do we measure θ?

Theorem. The angle θ between two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ satisfies

$$\vec{u} \cdot \vec{v} = \|\vec{u}\,\| \ \|\vec{u}\,\| \ \cos\theta.$$

Proof. The vectors $\vec{u}$, $\vec{v}$, and $\vec{v}-\vec{u}$ form the sides a triangle that lies in the subspace $\text{Span}\{\vec{u},\vec{v}\}$ (a plane, a line, or a point, all of which lie in a plane).

The law of cosines applies to this triangle:

$$\|\vec{v}-\vec{u}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\ \|\vec{v}\|\ \cos\theta.$$

It follows that

$$\begin{aligned}
\|\vec{u}\|\ \|\vec{v}\|\ \cos\theta &= \frac{1}{2}\big(\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{v}-\vec{u}\|^2\big) \\
&= \frac{1}{2}\big(\vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - (\vec{v}-\vec{u})\cdot(\vec{v}-\vec{u})\big) \\
&= \frac{1}{2}\big(\vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v} - \vec{v}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{u}\big) \\
&= \frac{1}{2}\big(\vec{v}\cdot\vec{u} + \vec{u}\cdot\vec{v}\big) \\
&= \frac{1}{2}\big(\vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{v}\big) \\
&= \vec{u}\cdot\vec{v}.
\end{aligned}$$

This gives the formula that θ satisfies. □

Definition. Two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are said to be **orthogonal** if the angle between them is $\pi/2$, which is to say that their inner product is 0, and we write $\vec{u}\perp\vec{v}$.

Remember the Pythagorean Theorem that says in a right-angle triangle, the square of the hypothenuse equals the sum of the squares of the other sides?

Theorem 2 (The Pythagorean Theorem). Two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are orthogonal if and only if $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$.

Proof. For two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, we have

$$\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v} = \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2.$$

By this *identity*, if $\vec{u}\perp\vec{v}$, then $\vec{u}\cdot\vec{v}=0$, and so $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$.

On the other hand, by the identity, if $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$, then $\vec{u}\cdot\vec{v}=0$, and so $\vec{u}\perp\vec{v}$. □

Example. In $\mathbb{R}^4$, let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} -2 \\ 3 \\ 8 \\ 1 \end{bmatrix}$, and $\vec{z} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$.

Although we can not sketch these vectors, we can use the inner product to get useful information about them.

The inner products of pairs of these vectors are

$$\begin{aligned}
\vec{x}\cdot\vec{y} &= -2+6-8+4 = 0, \\
\vec{x}\cdot\vec{z} &= 1+2+1+4 = 8, \\
\vec{y}\cdot\vec{z} &= -2+3-8+1 = -6.
\end{aligned}$$

So $\vec{x} \perp \vec{y}$, while $\vec{x}$ is not orthogonal to $\vec{z}$, and $\vec{y}$ is not orthogonal to $\vec{z}$.

The lengths of these vectors are

$$\begin{aligned}
\|\vec{x}\,\| &= \sqrt{1+4+1+16} = \sqrt{22},\\
\|\vec{y}\,\| &= \sqrt{4+9+64+1} = \sqrt{78},\\
\|\vec{z}\,\| &= \sqrt{1+1+1+1} = 2.
\end{aligned}$$

So $\vec{z}$ is the shortest of these three, while $\vec{y}$ is the longest.

Unit vectors in the direction of $\vec{x}$, $\vec{y}$, and $\vec{z}$ respectively are

$$\frac{1}{\|\vec{x}\|}\vec{x} = \begin{bmatrix} 1/\sqrt{22} \\ 2/\sqrt{22} \\ -1/\sqrt{22} \\ 4/\sqrt{22} \end{bmatrix}, \quad \frac{1}{\|\vec{y}\|}\vec{y} = \begin{bmatrix} -2/\sqrt{78} \\ 3/\sqrt{78} \\ 8/\sqrt{78} \\ 1/\sqrt{78} \end{bmatrix}, \quad \frac{1}{\|\vec{z}\|}\vec{z} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}.$$

Now since $\vec{x} \perp \vec{y}$, the Pythagorean Law should hold for $\vec{x}$ and $\vec{y}$. We can verify this:

$$\|\vec{x} + \vec{y}\,\|^2 = \|[-1,5,7,5]^T\|^2 = 1 + 25 + 49 + 25 = 100$$

while

$$\|\vec{x}\,\|^2 + \|\vec{y}\,\|^2 = 22 + 78 = 100.$$

Orthogonal Complements.

We extend the notion of orthogonality between two vectors to between a vector and a subspace of vectors.

A vector $\vec{z}$ in $\mathbb{R}^n$ is orthogonal to a subspace W of $\mathbb{R}^n$ if $\vec{z}$ is orthogonal to every vector of W.

The **orthogonal complement** of W, denoted by $W^\perp$, is the set of all vectors $\vec{z}$ in $\mathbb{R}^n$ that are orthogonal to W.

Example. Every line L through the origin in $\mathbb{R}^3$ is a one dimensional subspace.

The orthogonal complement of L is a plane W through the origin that makes a right angle with L.

The orthogonal complement of a plane W through the origin is a line L through the origin that makes a right angle with W.

Theorem. Let W be a subspace of $\mathbb{R}^n$. A vector $\vec{x}$ in $\mathbb{R}^n$ is orthogonal to W if and only if $\vec{x}$ is orthogonal to every vector in a spanning set for W. The orthogonal complement $W^\perp$ is a subspace of $\mathbb{R}^n$.

You will provide a proof of this theorem in the homework.

An $m \times n$ matrix A has several subspaces associated to it: its null space, its row space, and its column space.

Are any of these the orthogonal complement of others?

Theorem 3. For an $m \times n$ matrix A, there holds

$$(\text{Row}\, A)^{\perp} = \text{Nul}(A), \text{ and } (\text{Col}\, A)^{\perp} = \text{Nul}(A^T).$$

Proof. By the row-column rule for computing $A\vec{x}$, we see that for each $\vec{x}$ in $\text{Nul}(A)$ is orthogonal to each row of A.

Since the rows of A are a spanning set for $\text{Row}(A)$, we have that every $\vec{x}$ in $\text{Nul}(A)$ is orthogonal to $\text{Row}(A)$.

Conversely, if $\vec{x}$ is orthogonal to $\text{Row}(A)$, then $\vec{x}$ is orthogonal to each row of A, and hence $A\vec{x} = 0$.

Hence $\vec{x}$ is in $\text{Nul}(A)$.

This shows that $(\text{Row}A)^{\perp} = \text{Nul}(A)$.

This works for any matrix, so it work for A^T which gives $(\text{Row}A^T)^{\perp} = \text{Nul}(A^T)$.

Since $\text{Row}(A^T) = \text{Col}(A)$, we have that $(\text{Col}A)^{\perp} = \text{Nul}(A^T)$. $\square$

Example. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

The null space of A is a one dimensional subspace of $\mathbb{R}^3$:

$$\text{Nul}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

The row space of A is a two dimensional subspace of $\mathbb{R}^3$:

$$\text{Row}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

Notice that the basis vector for $\text{Nul}(A)$ is orthogonal to each basis vector for $\text{Row}(A)$, so indeed $(\text{Row}\, A)^{\perp} = \text{Nul}(A)$.

The null space of A^T is the zero dimensional subspace of $\mathbb{R}^2$:

$$\text{Nul}(A^T) = \{\vec{0}\}.$$

The column space of A is the two dimensional subspace of $\mathbb{R}^2$:

$$\text{Col}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}.$$

Notice that the zero vector is orthogonal to every vector in $\text{Col}(A)$, so indeed $(\text{Col}\, A)^{\perp} = \text{Nul}(A^T)$.