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Math 313
Exam 3
June 9,10, 2015

Name: Key

Section:

Instructor:

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Instructions

I) Do not write on the barcode area at the top of each page, or near the four circles on each page.
II) Fill in the correct boxes for your BYU ID and for the correct answer on the multiple choice completely. Multiple choice questions are 4 points each.
III) Multiple choice questions that have more than one correct answer will be marked with a ♣. All other questions have only one correct answer.
IV) For questions which require a written answer, show all your work in the space provided and justify your answer.
V) Simplify your answers.
VI) Scientific calculators are allowed.
VII) No books or notes are allowed.
VIII) There is no time limit on this exam.

Part I: Multiple Choice Questions: *Questions marked with a ♣ have more than one correct answer. Mark **all** correct answers. The other questions have one right answer. (4 points each)*

1 ♣ Which of the following statements are correct? Mark all that apply.

- [x] A set S of p vectors that span a vector space space V of dimension p is a basis for V.
- [x] For each subspace H of a finite dimensional vector space V, we have $\dim H \leq \dim V$.
- [ ] The vector space $\mathbb{R}^2$ is a subspace of the vector space $\mathbb{R}^3$.
- [x] A linearly independent set S of n vectors in an n-dimensional vector space V is a basis for V.
- [x] The dimension of the vector space of all $m \times n$ matrices $M_{m\times n}$ is $m + n$.
- [ ] The vector space of all polynomials $\mathbb{P}$ is finite-dimensional.

2 What is the smallest possible dimension of $\operatorname{Nul} A$ for a 9×14 matrix A?

- [ ] 14
- [ ] 10
- [ ] 0
- [ ] 23
- [ ] 9
- [x] 5

3 ♣ Which of the following values of λ are not eigenvalues of $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & -1 & -1 \end{bmatrix}$? Mark all that apply.

- ☐ $\lambda = -2$
- ☒ $\lambda = -1$
- ☐ $\lambda = 2$
- ☐ $\lambda = 0$
- ☒ $\lambda = 1$
- ☒ $\lambda = 3$

$\lambda=-2: \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ $\lambda=-1: \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ $\lambda=2: \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -1 & -3 \end{bmatrix}$

$\lambda=1 \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & -1 & -2 \end{bmatrix}$

4 ♣ Which of the following matrices have $\lambda^2 - 10\lambda + 1$ as their characteristic polynomial? Mark all the apply.

- ☐ $\begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix}$
- ☐ $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$
- ☒ $\begin{bmatrix} 9 & 8 \\ 1 & 1 \end{bmatrix}$ $(9-\lambda)(1-\lambda) - 8 = \lambda^2 - 10\lambda + 1$
- ☐ $\begin{bmatrix} 0 & 1 \\ 1 & 5 \end{bmatrix}$
- ☐ $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
- ☒ $\begin{bmatrix} 8 & 5 \\ 3 & 2 \end{bmatrix}$ $(8-\lambda)(2-\lambda) - 15 = \lambda^2 - 10\lambda + 1$

5 ♣ Which of the following matrices are diagonalizable? Mark all that apply.

- [x] $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $(1-\lambda)^2-1=\lambda^2-2\lambda$ $\lambda=0,2$
- [ ] $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ $\lambda=2$: $\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ 1 free var
- [x] $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- [x] $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $(2-\lambda)(1-\lambda)-1=\lambda^2-3\lambda+1$ $\frac{+3\pm\sqrt{9-4}}{2}$
- [ ] $\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ $(1-\lambda)(3-\lambda)+1=\lambda^2-4\lambda+4$ $\lambda=2$ $\left[\begin{array}{cc|c} -1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right]$ 1 free var
- [x] $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

6 Find the eigenvalues of

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

- [ ] $\lambda = 2 \pm i$
- [ ] $\lambda = -2 \pm i$
- [ ] $\lambda = -1, 3$
- [x] $\lambda = 1 \pm 2i$
- [ ] $\lambda = -1 \pm 2i$
- [ ] $\lambda = 1, -3$

$(1-\lambda)(1-\lambda)+4=\lambda^2-2\lambda+5$

$\lambda=\frac{2\pm\sqrt{4-20}}{2}$

7 ♣ Which vectors are orthogonal to $\mathbf{w} = (\mathbf{1}, \mathbf{1}, \mathbf{2}, -\mathbf{2})^{\mathbf{T}}$? Mark all that apply.

- [ ] $(2, -2, -1, 1)^T$
- [ ] $(3, -6, 2, -2)^T$
- [ ] $(2, 2, -4, 6)^T$
- [x] $(0, 0, 0, 0)^T$
- [ ] $(3, 1, 2, 3)^T$
- [ ] $(1, 2, 1, -2)^T$

8 Find the projection of $(1, 2, -2, 1)^T$ onto the vector $(0, -1, 1, 3)^T$.

- [ ] $\frac{-1}{11}(1, 2, -2, 1)^T$
- [ ] $\frac{1}{10}(1, 2, -2, 1)^T$
- [ ] $\frac{1}{10}(0, -1, 1, 3)^T$
- [x] $\frac{-1}{11}(0, -1, 1, 3)^T$
- [ ] $\frac{1}{11}(0, -1, 1, 3)^T$
- [ ] $\frac{-1}{10}(1, 2, -2, 1)^T$

$(1,2,-2,1) \cdot (0,-1,1,3) = -1$

$(0,-1,1,3) \cdot (0,-1,1,3) = 11$

$\frac{-1}{11}(0,-1,1,3)$

9 Let $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{w}$ be given as below. Find the orthogonal projection of $\mathbf{w}$ onto $\mathrm{Span}(\mathbf{v_1},\ \mathbf{v_2})$.

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix},\ \mathbf{v_2} = \begin{bmatrix} 0 \\ 2 \\ -1 \\ -2 \end{bmatrix},\ \mathbf{w} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

- ☐ $(2, 14, -24, -8)^T$
- ☒ $(4/3, 28/9, -2/9, 20/9)^T$
- ☐ $(12, 22, -4, 22)^T$
- ☐ $(1/3, 2, -8/3, -2/3)^T$
- ☐ $(4/3, 7/3, 2/3, 3)^T$

$\frac{12}{9}\begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix} + \frac{2}{9}\begin{bmatrix} 0 \\ 2 \\ -1 \\ -2 \end{bmatrix} =$

10 ♣ Which of the following statements are true? Mark all that apply.

- ☐ There are subspaces of $\mathbb{R}^n$ that do not have an orthogonal complement.
- ☐ The orthogonal complement of the column space of a matrix is the null space of that matrix.
- ☒ If a subspace of $\mathbb{R}^n$ has an orthogonal basis, it also has an orthonormal basis.
- ☒ Multiplying two vectors by an orthogonal matrix preserves their angle.
- ☐ An orthogonal matrix is a matrix with orthogonal columns.
- ☒ Multiplication of a vector by an orthogonal matrix preserves magnitude of the vector.

Part II: *Justify your answer and show all work for full credit.*

11 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8 ☐9 ☐10 *Administrative Use Only*

The scalar $\lambda = 1$ is an eigenvalue of $A = \begin{bmatrix} 2 & -8 & 2 \\ 2 & -15 & 4 \\ 8 & -64 & 17 \end{bmatrix}$. Find the eigenspace of A corresponding to the eigenvalue $\lambda = 1$.

$$A - 1I = \begin{bmatrix} 1 & -8 & 2 \\ 2 & -16 & 4 \\ 8 & -64 & 16 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -8 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - 8x_2 + 2x_3 = 0$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8t - 2s \\ t \\ s \end{bmatrix}$$

$$= t\begin{bmatrix} 8 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Eigenspace $= \text{Span}\left\{ \begin{bmatrix} 8 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

12 □0 □1 □2 □3 □4 □5 □6 □7 □8 □9 □10 *Administrative Use Only*

Show that the polynomials $p_1(t) = 1$, $p_2(t) = 2t$, $p_3(t) = -2+4t^2$, and $p_3(t) = -12t+8t^3$ form a basis for the vector space $\mathbb{P}_3$. [Hint; use the coordinate mapping corresponding to the basis $\{1, t, t^2, t^3\}$.]

$$[1] = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [2t] = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$[-2+4t^2] = \begin{bmatrix} -2 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \quad [-12t+8t^3] = \begin{bmatrix} 0 \\ -12 \\ 0 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix} \sim I$$

Since the coordinate vectors are linear independent, and $\dim \mathbb{P}_3 = 4$, the polynomials form a basis of $\mathbb{P}_3$.

12 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8 ☐9 ☐10 *Administrative Use Only*

Show that the polynomials $p_1(t) = 1$, $p_2(t) = 2t$, $p_3(t) = -2+4t^2$, and $p_3(t) = -12t+8t^3$ form a basis for the vector space $\mathbb{P}_3$. [Hint; use the coordinate mapping corresponding to the basis $\{1, t, t^2, t^3\}$.]

$$[1] = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [2t] = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$[-2+4t^2] = \begin{bmatrix} -2 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \quad [-12t+8t^3] = \begin{bmatrix} 0 \\ -12 \\ 0 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix} \sim I$$

Since the coordinate vectors are linear independent, and $\dim \mathbb{P}_3 = 4$, the polynomials form a basis of $\mathbb{P}_3$.

14 □0 □1 □2 □3 □4 □5 □6 □7 □8 □9 □10 *Administrative Use Only*

Let

$$K = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Find the eigenvalues and eigenvectors of K knowing that $\lambda = 1$ is one of the eigenvalues.

$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 0 & 1-\lambda \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 + (1-\lambda) = (1-\lambda)\left[(1-\lambda)^2+1\right]$$

$$= (1-\lambda)\left[\lambda^2-2\lambda+2\right] \quad \lambda=1, \lambda = \frac{2\pm\sqrt{4-8}}{2} = 1\pm i$$

$\lambda=1$:

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right] \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$\lambda = 1+i$

$$\left[\begin{array}{ccc|c} -i & 0 & 1 & 0 \\ 0 & -i & 0 & 0 \\ -1 & 0 & -i & 0 \end{array}\right] \begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix}$$

$\lambda = 1-i$

$$\begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}$$

15 ☐0 ☐1 ☐2 ☐3 ☐4 ☐5 ☐6 ☐7 ☐8 ☐9 ☐10 *Administrative Use Only*

Let

$$\mathbf{v_1} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \ \mathbf{v_2} = \begin{bmatrix} -2/3 \\ 2/3 \\ -1/3 \end{bmatrix}, \ \mathbf{w} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}.$$

Find the distance from $\mathbf{w}$ to the plane spanned by $\mathbf{v_1}$ and $\mathbf{v_2}$.

$$\text{Proj } \underline{w} = \frac{1}{3}\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} - \frac{11}{3}\begin{bmatrix} -2/3 \\ 2/3 \\ -1/3 \end{bmatrix} = \begin{bmatrix} 23/9 \\ -20/9 \\ 13/9 \end{bmatrix}$$

$$\underline{w} - \text{proj } \underline{w} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} 23/9 \\ -20/9 \\ 13/9 \end{bmatrix} = \begin{bmatrix} 4/9 \\ 2/9 \\ -4/9 \end{bmatrix}$$

$$\|\underline{w} - \text{proj } \underline{w}\| = \frac{\sqrt{16+4+16}}{9} = \frac{\sqrt{36}}{9} = \frac{6}{9} = \frac{2}{3}$$

16 □0 □1 □2 □3 □4 □5 *Administrative Use Only*

If A is a 3×3 matrix for which $A^T = -A$, prove that 0 is an eigenvalue of A.

$\det(A) = \det(A^T) = \det(-A) = (-1)^3 \det(A) = -\det(A)$

Thus $\det(A) = 0$

17 □0 □1 □2 □3 □4 □5 *Administrative Use Only*

Show for all vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$ that $\|\mathbf{u} + \mathbf{v}\|^{\mathbf{2}} - \|\mathbf{u} - \mathbf{v}\|^{\mathbf{2}} = \mathbf{4u} \cdot \mathbf{v}$.

$\|\underline{u}+\underline{v}\|^2 - \|\underline{u}-\underline{v}\|^2 = (\underline{u}+\underline{v})\cdot(\underline{u}+\underline{v}) - (\underline{u}-\underline{v})\cdot(\underline{u}-\underline{v}) = \cancel{\underline{u}\cdot\underline{u}} + 2\underline{u}\cdot\underline{v} + \cancel{\underline{v}\cdot\underline{v}} - (\cancel{\underline{u}\cdot\underline{u}} - 2\underline{u}\cdot\underline{v} + \cancel{\underline{v}\cdot\underline{v}})$

$= 4\,\underline{u}\cdot\underline{v}$