



NS



001

**Math 313**  
**Exam 3**  
June 9,10, 2015

Name: Key

Section: \_\_\_\_\_

Instructor: \_\_\_\_\_

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**Instructions**

- I) Do not write on the barcode area at the top of each page, or near the four circles on each page.
- II) Fill in the correct boxes for your BYU ID and for the correct answer on the multiple choice completely. Multiple choice questions are 4 points each.
- III) Multiple choice questions that have more than one correct answer will be marked with a ♣. All other questions have only one correct answer.
- IV) For questions which require a written answer, show all your work in the space provided and justify your answer.
- V) Simplify your answers.
- VI) Scientific calculators are allowed.
- VII) No books or notes are allowed.
- VIII) There is no time limit on this exam.



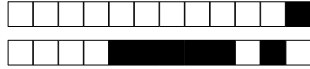
**Part I: Multiple Choice Questions:** Questions marked with a ♣ have more than one correct answer. Mark **all** correct answers. The other questions have one right answer. (4 points each)

1 ♣ Which of the following statements are correct? Mark all that apply.

- A set  $S$  of  $p$  vectors that span a vector space  $V$  of dimension  $p$  is a basis for  $V$ .
- For each subspace  $H$  of a finite dimensional vector space  $V$ , we have  $\dim H \leq \dim V$ .
- The vector space  $\mathbb{R}^2$  is a subspace of the vector space  $\mathbb{R}^3$ .
- A linearly independent set  $S$  of  $n$  vectors in an  $n$ -dimensional vector space  $V$  is a basis for  $V$ .
- The dimension of the vector space of all  $m \times n$  matrices  $M_{m \times n}$  is  $m + n$ .
- The vector space of all polynomials  $\mathbb{P}$  is finite-dimensional.

2 What is the smallest possible dimension of  $\text{Nul } A$  for a  $9 \times 14$  matrix  $A$ ?

- 14
- 10
- 0
- 23
- 9
- 5



3 ♣ Which of the following values of  $\lambda$  are not eigenvalues of  $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & -1 & -1 \end{bmatrix}$ ?

Mark all that apply.

$\lambda = -2$

$$\lambda = -2: \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\lambda = -1: \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda = 2: \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -1 & -3 \end{bmatrix}$$

$\lambda = -1$

$\lambda = 2$

$\lambda = 0$

$$\lambda = 1: \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & -1 & -2 \end{bmatrix}$$

$\lambda = 1$

$\lambda = 3$

4 ♣ Which of the following matrices have  $\lambda^2 - 10\lambda + 1$  as their characteristic polynomial? Mark all the apply.

$\begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

$\begin{bmatrix} 9 & 8 \\ 1 & 1 \end{bmatrix}$

$$(9-\lambda)(1-\lambda) - 8 = \lambda^2 - 10\lambda + 1$$

$\begin{bmatrix} 0 & 1 \\ 1 & 5 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\begin{bmatrix} 8 & 5 \\ 3 & 2 \end{bmatrix}$

$$(8-\lambda)(2-\lambda) - 15 = \lambda^2 - 10\lambda + 1$$



5 ♣ Which of the following matrices are diagonalizable? Mark all that apply.

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$   $(1-\lambda)^2 - 1 = \lambda^2 - 2\lambda$   $\lambda = 0, 2$

$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$   $\lambda = 2: \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  1 free var

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$   $(2-\lambda)(1-\lambda) - 1 = \lambda^2 - 3\lambda + 1$   $\frac{+3 \pm \sqrt{9-4}}{2}$

$\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$   $(1-\lambda)(3-\lambda) + 1 = \lambda^2 - 4\lambda + 4$   $\lambda = 2$   $\begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$  1 free var

$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

6 Find the eigenvalues of

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

$\lambda = 2 \pm i$

$\lambda = -2 \pm i$

$\lambda = -1, 3$

$\lambda = 1 \pm 2i$

$\lambda = -1 \pm 2i$

$\lambda = 1, -3$

$$(1-\lambda)(1-\lambda) + 4 = \lambda^2 - 2\lambda + 5$$

$$\lambda = \frac{2 \pm \sqrt{4-20}}{2}$$



7 ♣ Which vectors are orthogonal to  $\mathbf{w} = (1, 1, 2, -2)^T$ ? Mark all that apply.

- $(2, -2, -1, 1)^T$
- $(3, -6, 2, -2)^T$
- $(2, 2, -4, 6)^T$
- $(0, 0, 0, 0)^T$
- $(3, 1, 2, 3)^T$
- $(1, 2, 1, -2)^T$

8 Find the projection of  $(1, 2, -2, 1)^T$  onto the vector  $(0, -1, 1, 3)^T$ .

- $\frac{-1}{11}(1, 2, -2, 1)^T$
- $\frac{1}{10}(1, 2, -2, 1)^T$
- $\frac{1}{10}(0, -1, 1, 3)^T$
- $\frac{-1}{11}(0, -1, 1, 3)^T$
- $\frac{1}{11}(0, -1, 1, 3)^T$
- $\frac{-1}{10}(1, 2, -2, 1)^T$

$$(1, 2, -2, 1) \cdot (0, -1, 1, 3) = -1$$

$$(0, -1, 1, 3) \cdot (0, -1, 1, 3) = 11$$

$$\frac{-1}{11}(0, -1, 1, 3)$$



9 Let  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ , and  $\mathbf{w}$  be given as below. Find the orthogonal projection of  $\mathbf{w}$  onto  $\text{Span}(\mathbf{v}_1, \mathbf{v}_2)$ .

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

$(2, 14, -24, -8)^T$

$(4/3, 28/9, -2/9, 20/9)^T$

$$\frac{12}{9} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix} + \frac{2}{9} \begin{bmatrix} 0 \\ 2 \\ -1 \\ -2 \end{bmatrix} =$$

$(12, 22, -4, 22)^T$

$(1/3, 2, -8/3, -2/3)^T$

$(4/3, 7/3, 2/3, 3)^T$

10 ♣ Which of the following statements are true? Mark all that apply.

There are subspaces of  $\mathbb{R}^n$  that do not have an orthogonal complement.

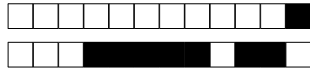
The orthogonal complement of the column space of a matrix is the null space of that matrix.

If a subspace of  $\mathbb{R}^n$  has an orthogonal basis, it also has an orthonormal basis.

Multiplying two vectors by an orthogonal matrix preserves their angle.

An orthogonal matrix is a matrix with orthogonal columns.

Multiplication of a vector by an orthogonal matrix preserves magnitude of the vector.



Part II: Justify your answer and show all work for full credit.

11  0  1  2  3  4  5  6  7  8  9  10 Administrative Use Only

The scalar  $\lambda = 1$  is an eigenvalue of  $A = \begin{bmatrix} 2 & -8 & 2 \\ 2 & -15 & 4 \\ 8 & -64 & 17 \end{bmatrix}$ . Find the eigenspace of  $A$  corresponding to the eigenvalue  $\lambda = 1$ .

$$A - 1I = \begin{bmatrix} 1 & -8 & 2 \\ 2 & -16 & 4 \\ 8 & -64 & 16 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -8 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - 8x_2 + 2x_3 = 0$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8t - 2s \\ t \\ s \end{bmatrix}$$

$$= t \begin{bmatrix} 8 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Eigenspace

$$= \text{Span} \left\{ \begin{bmatrix} 8 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$



12

 0  1  2  3  4  5  6  7  8  9  10 *Administrative Use Only*

Show that the polynomials  $p_1(t) = 1$ ,  $p_2(t) = 2t$ ,  $p_3(t) = -2 + 4t^2$ , and  $p_4(t) = -12t + 8t^3$  form a basis for the vector space  $\mathbb{P}_3$ . [Hint; use the coordinate mapping corresponding to the basis  $\{1, t, t^2, t^3\}$ .]

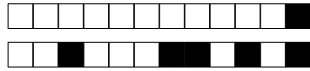
$$[1] = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [2t] = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$[-2 + 4t^2] = \begin{bmatrix} -2 \\ 0 \\ 4 \\ 0 \end{bmatrix}, [-12t + 8t^3] = \begin{bmatrix} 0 \\ -12 \\ 0 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix} \sim I$$

Since the coordinate vectors are linear independent, and  $\dim \mathbb{P}_3 = 4$ , the polynomials form a basis of  $\mathbb{P}_3$ .





12

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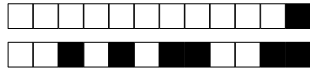
Show that the polynomials  $p_1(t) = 1$ ,  $p_2(t) = 2t$ ,  $p_3(t) = -2 + 4t^2$ , and  $p_4(t) = -12t + 8t^3$  form a basis for the vector space  $\mathbb{P}_3$ . [Hint; use the coordinate mapping corresponding to the basis  $\{1, t, t^2, t^3\}$ .]

$$[1] = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [2t] = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix} \sim I$$

Since the coordinate vectors are linear independent, and  $\dim \mathbb{P}_3 = 4$ , the polynomials form a basis of  $\mathbb{P}_3$ .

14  0  1  2  3  4  5  6  7  8  9  10 *Administrative Use Only*

Let

$$K = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Find the eigenvalues and eigenvectors of  $K$  knowing that  $\lambda = 1$  is one of the eigenvalues.

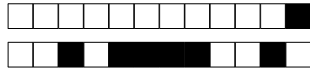
$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 0 & 1-\lambda \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 + (1-\lambda) = (1-\lambda)[(1-\lambda)^2 + 1]$$
$$= (1-\lambda)[\lambda^2 - 2\lambda + 2] \quad \lambda=1, \lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

 $\lambda=1$ :

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda=1+i$$
$$\begin{bmatrix} -i & 0 & 1 & 0 \\ 0 & -i & 0 & 0 \\ -1 & 0 & -i & 0 \end{bmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda=1-i$$
$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

15  0  1  2  3  4  5  6  7  8  9  10 *Administrative Use Only*

Let

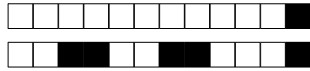
$$\mathbf{v}_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2/3 \\ 2/3 \\ -1/3 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}.$$

Find the distance from  $\mathbf{w}$  to the plane spanned by  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .

$$\text{Proj } \mathbf{w} = \frac{1}{3} \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} - \frac{11}{3} \begin{bmatrix} -2/3 \\ 2/3 \\ -1/3 \end{bmatrix} = \begin{bmatrix} 23/9 \\ -20/9 \\ 13/9 \end{bmatrix}$$

$$\mathbf{w} - \text{proj } \mathbf{w} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} 23/9 \\ -20/9 \\ 13/9 \end{bmatrix} = \begin{bmatrix} 4/9 \\ 2/9 \\ -4/9 \end{bmatrix}$$

$$\|\mathbf{w} - \text{proj } \mathbf{w}\| = \frac{\sqrt{16+4+16}}{9} = \frac{\sqrt{36}}{9} = \frac{6}{9} = \frac{2}{3}$$



16

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If  $A$  is a  $3 \times 3$  matrix for which  $A^T = -A$ , prove that 0 is an eigenvalue of  $A$ .

$$\det(A) = \det(A^T) = \det(-A) = (-1)^3 \det(A) = -\det(A)$$

$$\text{Thus } \det(A) = 0$$

17

 0  1  2  3  4  5 *Administrative Use Only*

Show for all vectors  $\mathbf{u}, \mathbf{v}$  in  $\mathbb{R}^n$  that  $\|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2 = 4\mathbf{u} \cdot \mathbf{v}$ .

$$\|\underline{u} + \underline{v}\|^2 - \|\underline{u} - \underline{v}\|^2 = (\underline{u} + \underline{v}) \cdot (\underline{u} + \underline{v}) - (\underline{u} - \underline{v}) \cdot (\underline{u} - \underline{v}) = \underline{u} \cdot \underline{u} + 2\underline{u} \cdot \underline{v} + \underline{v} \cdot \underline{v} - (\underline{u} \cdot \underline{u} - 2\underline{u} \cdot \underline{v} + \underline{v} \cdot \underline{v})$$

$$= 4\underline{u} \cdot \underline{v}$$