Math 313 (Linear Algebra)
Final Exam Practice KEY

Part I: Multiple Choice Questions: *Mark the correct answers for each question*

1. Find a basis of $W^\perp$ where W has basis $B = \{(1,1,1,1)^T, (1,2,0,0)^T\}$.

   Solution: We need to find all vectors $(a,b,c,d)^T$ whose dot product with the vectors above is 0. This leads to
$$a + b + c + d = 0,$$
   and
$$a + 2b = 0.$$
   In augmented matrix form we have
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 2 & 2 & 0 \\ 0 & 1 & -1 & -1 & 0 \end{bmatrix}$$
   We see that c and d are free variables and that $a = -2c - 2d$, $b = c + d$. Thus,
$$(a,b,c,d)^T = (-2c-2d, c+d, c, d)^T = c(-2,1,1,0)^T + d(-2,1,0,1)^T.$$

2. A certain experiment produces the data points $(4,8), (9,15)$, and $(16,24)$. These data points lie on a curve of the form
$$y = \beta_0 x + \beta_1 \sqrt{x}.$$
   The parameters β_0 and β_1 can be found by solving which of the following matrix equations:

   a) $\begin{bmatrix} 4 & 9 & 16 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 15 \\ 24 \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$

   b) $\begin{bmatrix} 4 & 2 \\ 9 & 3 \\ 16 & 4 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ 24 \end{bmatrix}$

   c) $\begin{bmatrix} 1 & 4 \\ 1 & 9 \\ 1 & 16 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ 24 \end{bmatrix}$

   d) $\begin{bmatrix} 1 & 8 \\ 1 & 15 \\ 1 & 24 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 16 \end{bmatrix}$

   Solution: b). We have that
$$\begin{bmatrix} 4 & 2 \\ 9 & 3 \\ 16 & 4 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \beta_0 \cdot 4 + \beta_1 \cdot \sqrt{4} \\ \beta_0 \cdot 9 + \beta_1 \cdot \sqrt{9} \\ \beta_0 \cdot 16 + \beta_1 \cdot \sqrt{16} \end{bmatrix}.$$
   In other words, the components of the vector above are the y-coordinates of the points on the curve with x-coordinates $4, 9$, and 16. Since the data points observed all lie on this curve, this vector should equal the vector consisting of all the y-coordinates, i.e.
$$\begin{bmatrix} 4 & 2 \\ 9 & 3 \\ 16 & 4 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \beta_0 \cdot 4 + \beta_1 \cdot \sqrt{4} \\ \beta_0 \cdot 9 + \beta_1 \cdot \sqrt{9} \\ \beta_0 \cdot 16 + \beta_1 \cdot \sqrt{16} \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ 24 \end{bmatrix}.$$

3. Which of the following is FALSE?

   a) Symmetric matrices are always diagonalizable.

   b) Symmetric matrices are always invertible.

   c) Eigenvectors of a symmetric matrix which come from different eigenspaces must be orthogonal.

   d) Eigenvectors of a symmetric matrix which come from different eigenspaces must be linearly independent.

Solution: b) is false (and hence the correct answer). The zero matrix is symmetric, but is never invertible. a) and c) are true by Theorems 1 and 2 in Section 7.1. d) is true by Theorem 2 of Section 5.1.

4. Select the matrix corresponding to the quadratic form

$$Q(x_1, x_2, x_3, x_4) = -2x_1^2 + 5x_2^2 - 7x_3^2 - 4x_4^2 - 6x_1x_2 + 4x_3x_1 - 10x_1x_4$$

a) $\begin{bmatrix} -2 & 5 & -7 & -4 \\ 2 & 3 & 4 & -10 \\ 5 & -7 & -4 & -6 \\ 2 & -3 & -5 & 0 \end{bmatrix}$

b) $\begin{bmatrix} -2 & -3 & 2 & -5 \\ -3 & 5 & 0 & 0 \\ 2 & 0 & -7 & 0 \\ -5 & 0 & 0 & -4 \end{bmatrix}$

c) $\begin{bmatrix} -2 & -6 & 4 & -10 \\ -6 & 5 & 0 & 0 \\ 4 & 0 & -7 & 0 \\ -10 & 0 & 0 & -4 \end{bmatrix}$

d) $\begin{bmatrix} -2 & -6 & 4 & -10 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$

Solution: b).

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5. Which of the sets is orthogonal under the given inner product on $C[0, \pi]$?

$$\langle f, g \rangle = \int_0^\pi f(x)g(x)dx$$

a) $\{1, \sin x\}$

b) $\{1, \cos x\}$

c) $\{1, -1\}$

d) $\{\sin x, \cos x\}$

Solution: b) and d). We can compute

$$\langle 1, \sin x \rangle = \int_0^\pi 1 \cdot \sin x \, \mathrm{d}x = -\cos \pi + \cos 0 = 2$$

$$\langle 1, \cos x \rangle = \int_0^\pi 1 \cdot \cos x \, \mathrm{d}x = \sin \pi - \sin 0 = 0$$

$$\langle 1, -1 \rangle = \int_0^\pi 1 \cdot (-1) \, \mathrm{d}x = -\pi + 0 = -\pi$$

$$\langle \cos x, \sin x \rangle = \int_0^\pi \cos x \cdot \sin x \, \mathrm{d}x = \tfrac{1}{2}(\cos \pi)^2 - \tfrac{1}{2}(\cos 0)^2 = 0$$

where the last integral is computed using a change of variables $u = \sin x$ and $\mathrm{d}u = \cos x \, \mathrm{d}x$.

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6. Let A be a symmetric matrix. Eigenvectors of A which correspond to *distinct* eigenvalues are always

   a) Linearly Independent

   b) Orthogonal

   c) Nonzero

   d) All of the above

   Solution: a), b), c) and d) are all correct. c) is correct by the definition of an eigenvector.

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7. Let $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ be a quadratic form, with A a symmetric matrix. Which statements are always true?

   a) Q is positive definite

   b) Introducing a new variable by setting $\mathbf{x} = P\mathbf{y}$ allows eliminating mixed terms.

   c) A is orthogonally diagonalizable.

   d) If $A = B^T B$ then A is positive semidefinite.

   e) If $A = B^T B$ then A is positive definite.

   Solution: b), c), and d) are true. Note that d) is true because if $A = B^T B$, then for any $\mathbf{x}$ we have

   $$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \mathbf{x}^T B^T B\mathbf{x} = (B\mathbf{x})^T (B\mathbf{x}) = (B\mathbf{x}) \cdot (B\mathbf{x}) = \|B\mathbf{x}\|^2 \geq 0.$$

   Notice this only proves that the quadratic form Q is positive semidefinite, because if $\mathbf{x}$ is a nontrivial element of Nul B, then $Q(\mathbf{x}) = \|B\mathbf{x}\|^2 = \|\mathbf{0}\|^2 = 0$, even though $\mathbf{x} \neq \mathbf{0}$.

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8. Let A and B be $n \times n$ matrices. Which of the following is FALSE?

   a) $\det(AB) = \det A \det B$

   b) $\det(A^T) = \det A$

   c) $\det(A^{-1}) = \det A$

   d) $\det(kA) = k^n \det A$

   Solution: c) is false. Indeed, if A is an invertible matrix then

   $$\det(A^{-1}) = \frac{1}{\det A}.$$

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9. What is the minimum value of $\mathbf{x}^T A\mathbf{x}$ subject to $\mathbf{x}^T\mathbf{x}$=1, if $A = \begin{bmatrix} 2 & 5 \\ 5 & 2 \end{bmatrix}$

a) 0 b) -3 c) 3 d) -7 e) 7

Solution: b). The eigenvalues of A are 7 and -3. The minimum value of the Q along the unit circle will be the smallest eigenvalue, i.e. $\lambda = -3$.

Part II: *Fill in the blank with the* ***best*** *possible answer. (x points each.)*

10. The Cauchy-Schwarz Inequality states that for all $\mathbf{u}, \mathbf{v}$ in a vector space V, $\underline{\quad |\langle \mathbf{u}, \mathbf{v} \rangle| \quad} \leq \underline{\quad \|\mathbf{u}\|\|\mathbf{v}\| \quad}$.

11. Let A, B, and C be invertible $n \times n$ matrices. Then the inverse of $AB^{-1}C^T A^{-1}B$ is equal to $\underline{\quad B^{-1}A(C^{-1})^T BA^{-1} \quad}$.

Part III: *Justify your answer and show all work for full credit.*

12. Find a basis of $W^{\perp}$ where W has basis $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \right\}$.

Solution: Let A be the matrix whose rows are the vectors in the basis B, i.e.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \end{bmatrix}.$$

Then $W = \text{Row } A$, and $W^{\perp} = (\text{Row } A)^{\perp} = \text{Nul } A$. Row reducing gives

$$\begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

and hence a basis for $\text{Nul } A = W^{\perp}$ is given by

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

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13. Find the equation of the least-squares line that best fits the data points $(-1, -2)$, $(0, 1)$, $(1, 1)$, $(2, 1)$, and $(3, 4)$.

Solution: The design matrix is given by

$$X = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

while we have $\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \\ 4 \end{bmatrix}$. The normal equations $X^T X \beta = X^T \mathbf{y}$ give

$$X^T X \beta = \begin{bmatrix} 5 & 5 \\ 5 & 15 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 5 \\ 17 \end{bmatrix} = X^T \mathbf{y}.$$

Solving gives $\beta = \begin{bmatrix} -1/5 \\ 6/5 \end{bmatrix}$. Thus the equation of the least squares line is

$$y = -\frac{1}{5} + \frac{6}{5} x.$$

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14. The matrix $A = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$ has eigenvalues -2 and 1. Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$.

Solution: The characteristic polynomial of A is given by

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 1 & -1 \\ 1 & -1-\lambda & -1 \\ -1 & -1 & -1-\lambda \end{vmatrix} = -\lambda^3 - 3\lambda^2 + 4.$$

By inspection we can guess that one root will be $\lambda_1 = 1$, and hence

$$\det(A - \lambda I) = (\lambda - 1)(-\lambda^2 - 4\lambda - 4) = -(\lambda - 1)(\lambda + 2)^2.$$

Thus the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = -2$. Finding a unit eigenvector for $\lambda_1 = 1$:

$$A - I = \begin{bmatrix} -2 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix}$$

which row reduces to

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and hence we get a corresponding unit eigenvector

$$\mathbf{u}_1 = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}.$$

Finding unit eigenvectors for $\lambda_2 = \lambda_3 = -2$:

$$A + 2I = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

which row reduces to

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and hence we get corresponding eigenvectors

$$\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Notice that these vectors are not orthogonal to each other (though both $\mathbf{x}_2$ and $\mathbf{x}_3$ are orthogonal to the other eigenvector $\mathbf{u}_1$, which is guaranteed because A is symmetric). Thus we perform Gram-Schmidt to the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ to make them orthogonal to each other:

$$\mathbf{v}_2 = \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{v}_2 \cdot \mathbf{x}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{-1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}.$$

Normalizing these vectors gives orthonormal eigenvectors

$$\mathbf{u}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{u}_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}.$$

Thus

$$P = \begin{bmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

We can easily check that $P^{-1} = P^T$ and that $A = PDP^T$.

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15. Let W be the subspace of $\mathbb{P}_2$ spanned by $\{t, t^2\}$. Find the orthogonal projection of 1 onto W using the inner product

$$\langle p, q\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1).$$

Solution: First we compute

$$\langle t, t^2\rangle = (-1)\cdot(-1)^2 + (0)\cdot(0)^2 + (1)\cdot(1)^2 = 0$$

and hence t and t^2 are orthogonal with respect to this inner product. Hence we can use this basis for W to compute the orthogonal projection of 1 to W (if they weren't orthogonal, we would have to use Gram-Schmidt to make them orthogonal to each other before proceeding). Then

$$\text{proj}_W\ 1 = \frac{\langle 1, t\rangle}{\langle t, t\rangle} t + \frac{\langle 1, t^2\rangle}{\langle t^2, t^2\rangle} t^2,$$

where

$$\begin{aligned} \langle t, t\rangle &= (-1)\cdot(-1) + (0)\cdot(0) + (1)\cdot(1) = 2 \\ \langle t^2, t^2\rangle &= (-1)^2\cdot(-1)^2 + (0)^2\cdot(0)^2 + (1)^2\cdot(1)^2 = 2 \\ \langle 1, t\rangle &= 1\cdot(-1) + 1\cdot(0) + 1\cdot(1) = 0 \\ \langle 1, t^2\rangle &= 1\cdot(-1)^2 + 1\cdot(0)^2 + 1\cdot(1)^2 = 2. \end{aligned}$$

Hence

$$\text{proj}_W\ 1 = \frac{0}{2} t + \frac{2}{2} t^2 = t^2.$$

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16. Prove that if λ is an eigenvalue of the matrix A, then λ^2 is an eigenvalue of the matrix A^2.

Solution: If λ is an eigenvalue of A, with corresponding eigenvector $\mathbf{v}$, then by definition $A\mathbf{v} = \lambda\mathbf{v}$. Multiplying this equation by A on both sides gives

$$\begin{aligned} A^2\mathbf{v} &= A(\lambda\mathbf{v}) \\ &= \lambda(A\mathbf{v}) \\ &= \lambda(\lambda\mathbf{v}) \\ &= \lambda^2\mathbf{v}. \end{aligned}$$

Thus $A^2\mathbf{v} = \lambda^2\mathbf{v}$, and hence λ^2 is an eigenvalue of A^2.

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17. Compute the eigenvalues and eigenvectors corresponding the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Solution: Computing the characteristic polynomial gives

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = (\lambda - i)(\lambda + i).$$

Thus the eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$. To compute the eigenvector associated to $\lambda_1 = i$, we look for a vector in Nul $(A - iI)$:

$$A - iI = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}.$$

We could try to row reduce this matrix to find a basis for it's null space, but we could also take a shortcut by observing that the first and second rows must be scalar multiples of each other (otherwise, the two rows would be linearly independent, and hence the matrix $A - iI$ would be invertible and have trivial null space). So we know we can cancel the second row using the first row (indeed the second row is $-i$ times the first row). Note that this shortcut only works for 2×2 matrices, any bigger and we'd have to do real row-reduction. Thus $-ix_1 + x_2 = 0$ and the eigenvector we're looking for is

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

We could do the same thing to find the second eigenvector, but we don't have to since it will be the complex conjugate of $\mathbf{v}_1$:

$$\mathbf{v_2} = \overline{\mathbf{v}}_\mathbf{1} = \overline{\begin{bmatrix} 1 \\ i \end{bmatrix}} = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

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18. Prove that similar matrices have the same determinant.

Solution: Let A and B be similar matrices. In other words, there is some invertible matrix P such that $A = PBP^{-1}$. Then

$$\det(A) = \det(PBP^{-1}) = \det(P)\det(B)\det(P^{-1}) = \det(P)\det(B)\frac{1}{\det(P)} = \det(B).$$

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19. Prove the Pythagorean Theorem: If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors in $\mathbb{R}^n$, then

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}|^2 + \|\mathbf{v}\|^2$$

Solution: Since $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, we have that $\mathbf{u} \cdot \mathbf{v} = 0$. Hence

$$\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

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20. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$. Note that $\mathcal{B}$ is a basis for $\mathbb{R}^2$. (You do not need to prove this.

(a) Find the change-of-coordinates matrix $P_{\mathcal{B}}$ from the basis $\mathcal{B}$ to the standard basis.

Solution: From Section 4.4 we have that

$$P_{\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

(b) What is the equation relating $\mathbf{x}$, $[\mathbf{x}]_{\mathcal{B}}$, and $P_{\mathcal{B}}$?

Solution: $\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$.

(c) Find the vector $\mathbf{x}$ if $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Solution:

$$\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}.$$

(d) Use an inverse matrix to find $[\mathbf{y}]_{\mathcal{B}}$ if $\mathbf{y} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Solution:

$$[\mathbf{y}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{y} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}.$$

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21. For given A and $\mathbf{b}$ compute all solutions $\hat{\mathbf{x}}$ of the least squares problem and the distance from $A\hat{\mathbf{x}}$ to $\mathbf{b}$.

(a) $A := \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{b} := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Solution: $A\mathbf{x} = \mathbf{b}$ is inconsistent, so we find the least squares approximation by solving the normal equation $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$, which gives $2\hat{\mathbf{x}} = 2$, and hence $\hat{\mathbf{x}} = 1$. The distance from $A\hat{\mathbf{x}}$ to $\mathbf{b}$ is given by

$$\left\| \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\| = 1.$$

(b) $A := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{b} := \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}$.

Solution: The normal equations of the system $A\mathbf{x} = \mathbf{b}$ give

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

and have solution

$$\hat{\mathbf{x}} = \begin{bmatrix} 5/3 \\ 2 \end{bmatrix}.$$

We have

$$\|A\hat{\mathbf{x}} - \mathbf{b}\| = \left\| \begin{bmatrix} 5/3 \\ 2 \\ 5/3 \\ 2 \\ 5/3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{bmatrix} \right\| = \sqrt{\frac{8}{3}}.$$

(c) $A := \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{b} := \begin{bmatrix} 3 \\ 6 \end{bmatrix}$.

Solution: The matrix A is invertible, hence the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$:

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Note that if you hadn't noticed that A was invertible, and still used the normal equations to solve for your least squares approximation, you still would have gotten the same answer. In this case the least squares approximation will be an exact solution.

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22. Find the inverse of the matrix

$$A = \begin{bmatrix} -3 & 4 & 0 \\ -2 & 3 & 0 \\ -2 & 2 & 1 \end{bmatrix}$$

using any method of your choosing.

Solution: One could either row reduce the augmented matrix $\left[\begin{array}{c|c} A & I \end{array}\right]$ to obtain $\left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$, or could use the adjugate formula. The matrix of cofactors is

$$\begin{bmatrix} 3 & 2 & 2 \\ -4 & -3 & -2 \\ 0 & 0 & -1 \end{bmatrix}.$$

Doing cofactor expansion of $\det A$ along the right hand column shows that $\det A = -1$. The transpose of the cofactor matrix is the adjugate of A:

$$\text{adj } A = \begin{bmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 2 & -2 & -1 \end{bmatrix},$$

and

$$A^{-1} = \frac{1}{\det A} \text{adj } A = \begin{bmatrix} -3 & 4 & 0 \\ -2 & 3 & 0 \\ -2 & 2 & 1 \end{bmatrix}.$$

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23. Let A, C, and D be $n \times n$ matrices, with $CA = I$ and $AD = I$. Prove that $C = D$.

Solution: Suppose that $CA = I$ and $AD = I$. Then

$$C = CI = C(AD) = (CA)D = ID = D.$$

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24. Let A be the transpose of $B := \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$. Find U, V, D and Σ for an SVD of A and prove its correctness.

Solution: We start by computing the eigenvalues and eigenvectors of

$$B^T B = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix},$$

which has characteristic polynomial

$$\det(B^T B - \lambda I) = \lambda(\lambda - 90),$$

and hence $\lambda_1 = 90$ and $\lambda_2 = 0$. Thus the singular values of B are $\sigma_1 = \sqrt{90}$ and $\sigma_2 = 0$, and

$$\Sigma = \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

We subtract λ_1 and λ_2 off the diagonal of $B^T B$ to find the corresponding eigenvectors, which are

$$\mathbf{v}_1 = \begin{bmatrix} -\frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix}.$$

Thus

$$V = \begin{bmatrix} -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}.$$

The columns of U can be computed

$$\mathbf{u}_1 = \frac{1}{\sigma_1} B\mathbf{v}_1 = \frac{1}{\sqrt{90}} \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} -\frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}.$$

The second and third columns of U can't be computed in this way, since we only have one nonzero singular value. We thus need to extend $\mathbf{u}_1$ to an orthonormal basis of $\mathbb{R}^3$. To find two vectors orthogonal to $\mathbf{u}_1$, place $\mathbf{u}_1$ as the row of a matrix A, and find a basis for Nul $A = (\text{Row } A)^\perp$. This gives us vectors

$$\mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

Note that both of these vectors are orthogonal to $\mathbf{u}_1$, but they are not orthogonal to each other (and are not unit vectors). So we apply Gram-Schmidt to these two vectors to give a pair of orthogonal vectors $\mathbf{y}_2$ and $\mathbf{y}_3$:

$$\mathbf{y}_2 = \mathbf{x}_2$$

$$\mathbf{y}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{y}_2}{\mathbf{y}_2 \cdot \mathbf{y}_2}\mathbf{y}_2 = \begin{bmatrix} \frac{2}{5} \\ -\frac{4}{5} \\ 1 \end{bmatrix}$$

Normalizing the vectors $\mathbf{y}_2$ and $\mathbf{y}_3$ give

$$\mathbf{u}_2 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{u}_3 = \begin{bmatrix} \frac{2}{3\sqrt{5}} \\ -\frac{4}{3\sqrt{5}} \\ \frac{\sqrt{5}}{3} \end{bmatrix},$$

and hence

$$U = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & -\frac{4}{3\sqrt{5}} \\ -\frac{2}{3} & 0 & \frac{\sqrt{5}}{3} \end{bmatrix}.$$

It is easy to verify that both V and U are orthogonal, and that $A = U\Sigma V^T$ as required.

END OF EXAM